Math 620: HW #1

1. Let $R$ be a Dedekind domain and let $I$ be a nonzero ideal of $R$. Suppose that $I^2$ and $I^3$ are principal ideals. Show that $I$ is principal.

2. Consider the equation $y^2 = x^3 - 13$, where $x$ and $y$ are integers.
   (a) Show that $x$ cannot be even.
   (b) Show that $13 \nmid y$.
   (c) Let $p$ be a prime ideal in $\mathbb{Z}[\sqrt{-13}]$ and suppose $p$ divides both $y + \sqrt{-13}$ and $y - \sqrt{-13}$. Show that $p$ divides either 2 or 13.
   (d) Show that if $p$ divides 2 then $x$ is even and if $p$ divides 13 then $13 \nmid y$.
   (e) Show that $y + \sqrt{-13}$ and $y - \sqrt{-13}$ are relatively prime in $\mathbb{Z}[\sqrt{-13}]$.
   (f) You may assume that the class number of $\mathbb{Z}[\sqrt{-13}]$ is 2 and that the only units are $\pm 1$. Show that $y + \sqrt{-13} = (a + b\sqrt{-13})^3$ for some integers $a$ and $b$.
   (g) Find all integer solutions of $y^2 = x^3 - 13$.

3. Let $R$ be a Dedekind domain and let $S \subset R$ be a subset closed under multiplication, with $0 \notin S$. Show that $S^{-1}R$ is a Dedekind domain.

4. Let $R$ be a Dedekind domain and let $P_1, \ldots, P_m$ be nonzero prime ideals of $R$. Let $S = R \setminus (P_1 \cup \cdots \cup P_m)$.
   (a) Show that $S$ is a multiplicatively closed subset of $R$.
   (b) Show that $S^{-1}R$ is a PID.

5. Let $R$ be a Dedekind domain. Let $\alpha$ and $\beta$ be nonzero elements of $R$. Show that the fraction $\alpha/\beta$ can be reduced (that is, $\alpha/\beta = a/b$ with $\gcd(a,b) = 1$) if and only if the ideal $(\alpha, \beta)$ is principal.

6. Let $R$ be a Dedekind domain and let $I$ and $J$ be nonzero ideals of $R$.
   (a) Show that there exists $j \in J$ such that $I + jJ^{-1} = R$, hence there exist $i \in I$, $j \in J$, $k \in J^{-1}$ such that $i + jk = 1$.
   (b) Let $M$ be the matrix $\begin{pmatrix} i & j \\ -k & 1 \end{pmatrix}$ and let $N = \begin{pmatrix} 1 & -j \\ k & i \end{pmatrix}$. Show that $(x, y) \mapsto (x, y)M$ maps $R \oplus IJ$ to $I \oplus J$, and that $(x, y) \mapsto (x, y)N$ is the inverse map. Therefore, $I \oplus J \simeq R \oplus IJ$ as $R$-modules.
   (c) Show that $I$ is a projective $R$-module.