Math 620: HW #1

1. Let R be a Dedekind domain and let I be a nonzero ideal of R. Suppose that I^2 and I^3 are principal ideals. Show that I is principal.

2. Consider the equation $y^2 = x^3 - 13$, where x and y are integers.
(a) Show that x cannot be even.
(b) Show that $13 \nmid y$.
(c) Let $\mathfrak{p}$ be a prime ideal in $\mathbb{Z}[\sqrt{-13}]$ and suppose $\mathfrak{p}$ divides both $y+\sqrt{-13}$ and $y - \sqrt{-13}$. Show that $\mathfrak{p}$ divides either 2 or 13.
(d) Show that if $\mathfrak{p}$ divides 2 then x is even and if $\mathfrak{p}$ divides 13 then $13 \mid y$.
(e) Show that $y + \sqrt{-13}$ and $y - \sqrt{-13}$ are relatively prime in $\mathbb{Z}[\sqrt{-13}]$.
(f) You may assume that the class number of $\mathbb{Z}[\sqrt{-13}]$ is 2 and that the only units are ± 1. Show that
$$y + \sqrt{-13} = (a + b\sqrt{-13})^3$$
for some integers a and b.
(g) Find all integer solutions of $y^2 = x^3 - 13$.

3. Let R be a Dedekind domain and let $S \subset R$ be a subset closed under multiplication, with $0 \notin S$. Show that $S^{-1}R$ is a Dedekind domain.

4. Let R be a Dedekind domain and let $P_1, \ldots, P_m$ be nonzero prime ideals of R. Let $S = R \setminus (P_1 \cup \cdots \cup P_m)$.
(a) Show that S is a multiplicatively closed subset of R.
(b) Show that $S^{-1}R$ is a PID.

5. Let R be a Dedekind domain. Let α and β be nonzero elements of R. Show that the fraction α/β can be reduced (that is, $\alpha/\beta = a/b$ with $\gcd(a, b) = 1$) if and only if the ideal (α, β) is principal.

6. Let R be a Dedekind domain and let I and J be nonzero ideals of R.
(a) Show that there exists $j \in J$ such that $I + jJ^{-1} = R$, hence there exist $i \in I$, $j \in J$, $k \in J^{-1}$ such that $i + jk = 1$.
(b) Let M be the matrix $\begin{pmatrix} i & j \\ -k & 1 \end{pmatrix}$ and let $N = \begin{pmatrix} 1 & -j \\ k & i \end{pmatrix}$. Show that $(x, y) \mapsto (x, y)M$ maps $R \oplus IJ$ to $I \oplus J$, and that $(x, y) \mapsto (x, y)N$ is the inverse map. Therefore, $I \oplus J \simeq R \oplus IJ$ as R-modules.
(c) Show that I is a projective R-module.