1. (a) Let G be a group. Suppose that every cyclic subgroup of G is normal in G. Show that every subgroup of G is normal in G.
(b) Let $H = \{\pm 1, \pm i, \pm j \pm k\}$, with $ij = k = -ji, i^2 = j^2 = k^2 = -1$, be the quaternion group with 8 elements. Let T be a 2-group. Show that every subgroup of $H \times T$ is normal in $H \times T$ if and only if $x^2 = e$ (=the identity of T) for all $x \in T$.
(c) Let A be an abelian group of odd order and let $G = H \times A$, , where H is the quaternion group as in part (b). Show that every subgroup of G is normal in G. (*Hint:* the Chinese remainder theorem implies that if a, b, c are integers with c odd, then $x \equiv a \pmod 4$, $x \equiv b \pmod c$ has a solution x.)

2. Let G be a finite nonabelian simple group (that is, G has no nontrivial normal subgroups). A subgroup H of G is called maximal if there are no subgroups K with $H \underset{\neq}{\subset} K \underset{\neq}{\subset} G$. The normalizer $N(H)$ of a subgroup H is the set of $g \in G$ such that $gHg^{-1} = H$. Suppose there is an abelian subgroup A of G such that A is also a maximal subgroup.
(a) Let $C \neq \{1\}$ be a subgroup of A. Show that $N(C) = A$.
(b) Suppose B is a maximal subgroup of G and that B is abelian. Show that either $A = B$ or $A \cap B = \{1\}$.
(c) Show that there are exactly $|G|/|A|$ distinct subgroups of G that are conjugate to A (that is, of the form gAg^{-1} with $g \in G$).
(d) Let $U = \cup_{g \in G} \quad gAg^{-1}$ be the union of all the conjugates of A. Show that $\frac{1}{2}\big(|G|-1\big) < |U|-1 < |G|-1$.
(e) Show that G must have a nonabelian maximal subgroup (Hints: clearly each element of G is contained in at least one maximal subgroup. Also, a conjugate of a maximal subgroup is maximal.)

3. Let G be a finite group such that, for each positive integer n dividing the order of G, there is at most one subgroup of G order n.
(a) Show that every subgroup of G is normal.
(b) Let p be the smallest prime dividing the order of G and let $z \in G$ be an element of order p. Show that z is in the center of G. (*Hint:* Construct a map $G \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$.)
(c) Show that if G is abelian then it must be cyclic.
(d) Show that G is abelian and therefore cyclic. (*Hint:* Use induction on the order of G to assume that all such groups of smaller order are cyclic.)

4. (a) Let G be a finite group of order $n = 2^k m$ with $k \geq 1$ and m odd. Recall that if $g \in G$, then left multiplication by g gives a permutation $\phi(g)$ of the n elements of G, and $\phi : G \to S_n$ is an injective homomorphism.
Suppose G contains an element g of order 2^k. Show that the permutation $\phi(g)$ is a product of m disjoint cycles, each of length 2^k.
(b) Let H be a subgroup of S_n of order greater than 2 and suppose H contains an odd permutation. Show that H has a nontrivial normal subgroup.
(c) Show that if a finite group G of even order greater than 2 has a cyclic Sylow 2-subgroup, then G has a nontrivial normal subgroup.

5. (a) Show that a finitely generated subgroup of the additive group $\mathbb{Q}$ of rational numbers is always cyclic.
(b) Show that a finitely generated subgroup of the quotient group $\mathbb{Q}/\mathbb{Z}$ (additive group of rationals modulo the integers) is always cyclic.

6. (a) Let p be a prime and suppose G is a group of order p^3. Let C be the center of G. Show that G/C is abelian.
(b) Let G be a group and let H be a subgroup of the center of G. Suppose G/H is isomorphic to $\mathbb{Q}$, the additive group of rational numbers. Show that G is abelian (Hint: you may use the result of problem 5(a)above).

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX