MATH 600: ABSTRACT ALGEBRA (L. WASHINGTON)
FINAL EXAM, DECEMBER 19, 1994

1. (20 points) Let G be a finite group of odd order and let N be a normal subgroup of order 17. Prove that N is contained in the center of G.

2. (25 points) Let H be a subgroup of a group G, and let $\phi : G \to H$ be a homomorphism. Suppose $\phi(h) = h$ for all $h \in H$. Let N be the kernel of ϕ.
(a) Prove that if G is abelian then $H \times N \simeq G$.
(b) Give an example that shows that the conclusion of (a) can be false if G is non-abelian.

3. (20 points) Let G be a group of order $555 = 3 \times 5 \times 37$.
(a) Show that G has a normal subgroup N of order 5×37.
(b) Determine how many Sylow 5-subgroups there are in G.

4. (20 points) Let R be an integral domain and assume R is Noetherian. Show that every element of R has at least one factorization into irreducibles.

5. (30 points) Let R be a commutative ring with 1. Let S be a non-empty subset of R that is closed under multiplication and does not contain 0. Let I be an ideal of R that is maximal (with respect to inclusion) among ideals that do not intersect S.
(a) Show that I exists.
(b) Show that I is a prime ideal.
(c) Show that the intersection of all prime ideals of R is precisely the set of nilpotent elements of R.

6. (25 points) Let R be a commutative ring with 1. An R-module M is called *cyclic* if it can be generated by one element.
(a) Prove that a cyclic module has the form R/I where I is an ideal of R.
(b) Assume R is a PID. Show that a submodule of a cyclic module is cyclic.
(c) Suppose R has the property that submodules of cyclic R-modules are always cyclic. Show that R is a PID.