Homework #9, Due Wednesday, Nov. 26

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1. Let R be a unique factorization domain.
(a) Show that every nonzero prime ideal of R contains a nonzero principal prime ideal.
(b) Let I be a nonzero principal prime ideal of R. Suppose J is a prime ideal such that $0 \subseteq J \subseteq I$. Show that either $J = 0$ or $J = I$.

2. Let R be a commutative ring with 1 and suppose R has a unique maximal ideal I. Let $x \in R$. Show that $x \in I$ if and only if $1 + ax$ is a unit of R for all $a \in R$.

3. Let R be a commutative ring with 1. The ring R is said to be Artinian if every descending chain of ideals stops. That is, for each sequence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

of ideals of R, there exists n such that $I_n = I_{n+1} = \cdots$.
(a) Show that if R is Artinian and I is an ideal of R, then R/I is an Artinian ring.
(b) Show that if R is an Artinian integral domain, then R is a field. (*Hint:* Consider the ideals $(x) \supseteq (x^2) \supseteq (x^3) \cdots$.)
(c) Show that if R is Artinian and P is a prime ideal of R, then P is a maximal ideal.
(d) Show that if $P_1, \ldots, P_n$ are distinct prime ideals of an Artinian ring R, then there exists an element $x \in P_1 \cap \cdots \cap P_{n-1}$ with $x \notin P_n$.
(e) Show that an Artinian ring has only finitely many prime ideals.

4. (a) Let R be a unique factorization domain. Suppose $a, b \in R$ are nonzero elements such that $a^3 = b^2$. Show that there is an element $c \in R$ such that $ac = b$.
(b) Show that there do not exist polynomials $g(X,Y), h(X,Y) \in \mathbb{C}[X,Y]$ such that $X \cdot g(X,Y) = Y + (X^3 - Y^2) \cdot h(X,Y)$.
(c) Show that $\mathbb{C}[X,Y]/(X^3 - Y^2)$ is not a unique factorization domain.