MATH 601 (Washington) **Exam 1** **March 17, 2004**

Use 5 pages. Do a separate problem on each page. Write your name on each page. Do not staple.

1. (15 points) Let M be the matrix over the complex numbers

$$\begin{pmatrix} 3 & 0 & 0 \\ a & 3 & 0 \\ b & c & 2 \end{pmatrix}.$$

Determine all possible minimal polynomials for M and give an example of values for a, b, c for each possibility.

2. (30 points = 15+15) In this problem, all modules are $\mathbb{Z}$-modules. Let $n \geq 1$ be an integer. Define

$$f : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}, \qquad f(x) = \frac{x}{n} \mod \mathbb{Z}$$

and

$$g : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \qquad g(y) = ny.$$

Then $0 \to \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact (you do not need to prove this).
(a) Let M be a $\mathbb{Z}$-module and let $H = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$. Use the long exact Ext sequence attached to the above exact sequence to show that

$$\mathrm{Ext}^1(M, \mathbb{Z}_n) \simeq H/nH.$$

(b) Let M be a projective $\mathbb{Z}$-module. Show that $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is a divisible group. (*Note:* this can be done directly, or by using part (a).)

3. (30 points = 15+15) (a) Let R be an integral domain and let $0 \neq r \in R$. Assume r is not a unit of R. Find an exact sequence $0 \to A \to B \to C \to 0$ of R-modules (depending on r) such that the induced sequence

$$0 \to A \otimes (R/rR) \to B \otimes (R/rR) \to C \otimes (R/rR) \to 0$$

is not exact (where $\otimes$ denotes $\otimes_R$).
(b) Let R be a PID and let M be a finitely generated R-module. Show that M is flat if and only if M is torsion-free (torsion-free means that whenever $0 \neq r \in R$ and $0 \neq m \in M$, then $rm \neq 0$).

4. (10 points) Let R be a commutative ring with 1, let I be an ideal in R, and let M be an R-module. Show that

$$\mathrm{Hom}_R(R/I,\, M) \simeq M[I] = \{m \in M \mid im = 0 \text{ for all } i \in I\}.$$

5. (15 points) Consider the following commutative diagram of modules (over some ring):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & {\scriptstyle h_1}\big\downarrow & & {\scriptstyle h_2}\big\downarrow & & {\scriptstyle h_3}\big\downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

Assume the rows are exact. Show that if h_1 and h_3 are injective, then h_2 is injective.