MATH 601 (Washington) Exam 1 Solutions March 17, 2004

1. The characteristic polynomial is $(X-3)^2(X-2)$ so the minimal polynomial is either $(X-3)(X-2)$, which happens for $a=b=c=0$, or $(X-3)^2(X-2)$, which happens for $a=1, b=c=0$.

2. (a) Part of the long exact sequence is

$$\cdots \to \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(M,\mathbb{Z}_n) \to \mathrm{Ext}^1(M,\mathbb{Q}/\mathbb{Z}) \to \cdots .$$

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, it is injective, so $\mathrm{Ext}^1(M,\mathbb{Q}/\mathbb{Z})=0$. Therefore we obtain

$$H \to H \to \mathrm{Ext}^1(M,\mathbb{Z}_n) \to 0,$$

where the map $H \to H$ is induced by g, hence is multiplication by n. This yields the result.

(b) Since M is projective, $\mathrm{Ext}^1(M,\mathbb{Z}_n)=0$ for all $n \geq 1$. From (b), $H = nH$ for all such n, so H is divisible. Without using (a), we can consider $h \in H$. This yields a diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle h} & & \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow{\;g\;} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

The definition of projective gives a map h_1 from H to the first $\mathbb{Q}/\mathbb{Z}$ that makes the diagram commute, and it is easy to see that $nh_1 = h$. This implies divisibility.

3. (a) Consider the exact sequence $0 \to R \to R \to R/rR \to 0$, where the map $R \to R$ is multiplication by r. Tensoring with R/rR yields the sequence

$$0 \to R/rR \to R/rR \to (R/rR)\otimes(R/rR) \to 0,$$

where the map $R/rR \to R/rR$ is multiplication by r. Clearly this is not injective, so the sequence is not exact.

(b) From the structure theorem for finitely generated modules over PID's, M is isomorphic to $R^n \oplus T$, where T is torsion. If M is torsion-free, then $T=0$, so M is free, hence flat. Conversely, suppose M is flat. Let $0 \neq r \in R$ and consider the exact sequence in (a). Tensoring with M preserves exactness, so

$$0 \to M\otimes R \to M \otimes R \to M \otimes (R/rR) \to 0$$

is exact. But the first map is $M \to M$, given by multiplication by r. Since it is injective, and since this holds for all $r \neq 0$, there cannot be any torsion in M.

4. Let $\phi \in \mathrm{Hom}_R(R/I, M)$. Map ϕ to $\phi(1+I)$. Let $i \in I$. Since $\phi(i+I)=0$, we have $i\phi(1+I) = \phi(i+I) = 0$. Therefore, $\phi(1+I) \in M[I]$. Since $\phi(1)$ determines ϕ, we have an injective map $\mathrm{Hom}_R(R/I,M) \to M[I]$.

Now let $m \in M[I]$. Define $\phi(r+I) = rm$. If $r_1 + I = r_2 + I$, then $r_1 - r_2 = i \in I$, so $r_1 m - r_2 m = im = 0$. Therefore, ϕ is well-defined, hence is in $\mathrm{Hom}_R(R/I, M)$. This shows that the map above is surjective.

5. The snake lemma yields an exact sequence

$$0 \to \mathrm{Ker}(h_1) \to \mathrm{Ker}(h_2) \to \mathrm{Ker}(h_3) \to \cdots .$$

If h_1 and h_3 are injective, then we obtain

$$0 \to 0 \to \mathrm{Ker}(h_2) \to 0 \to \cdots .$$

Therefore, $\mathrm{Ker}(h_2) = 0$.