Homework #10

1. Let G be a finite group and let $\rho : G \to GL_n(\mathbb{C})$ be an irreducible representation. Suppose ρ is injective. Show that the center of G is cyclic. (*Hint:* What do you know about finite subgroups of the multiplicative group of a field?)

2. Let G be a group and suppose there exists a subgroup $A \subseteq G$ with $[G : A] = d$. Let $\rho : G \to GL(V)$ be a representation of G. Restrict ρ to A and decompose this representation of A into irreducible representations. Let $\phi : A \to GL(W)$ be one of these irreducible representations of A.
(a) Let $g_1, \ldots, g_d$ be left coset representatives for A in G, so $G = \cup g_i A$. Let

$$V_1 = g_1 W + g_2 W + \cdots + g_d W \subseteq V.$$

Show that $\rho(G)$ maps V_1 into itself, so V_1 gives a subrepresentation V.
(b) Suppose ρ is irreducible. Show that $V_1 = V$.
(c) Suppose A is abelian. Show that every irreducible representation of G has dimension less than or equal to d.
(d) Let D be a dihedral group D_n. Show that every irreducible representation of D has dimension at most 2.

3. Find the character table of the quaternion group Q_8.

4. Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a two-dimensional complex repre sentation of the finite group G. Let V be the 4-dimensional vector space of 2×2 complex matrices, and let G act on V by

$$\tilde{\rho}(g)(M) = \rho(g) M \rho(g)^{-1}$$

for $M \in V$.
(a) Show that if ρ is irreducible then $\tilde{\rho}$ contains the trivial representation exactly once.
(b) Show that if ρ is the sum of two distinct one-dimensional representations, then $\tilde{\rho}$ contains the trivial representation exactly twice.
(c) Show that if ρ is the sum of two equal one-dimensional representations, then $\tilde{\rho}$ equals the sum of four copies of the trivia l representation.

5. Let G be a finite group and let H be a normal subgroup. Let $R = \mathbb{C}[G/H]$ be the group ring of G/H with complex coefficients. Then R is a complex vector space. For $\sigma \in G$, let T_σ be the linear transformation of R given by multiplication on the left by σ (so $T_\sigma(gH) = \sigma g H$). Define a representation ρ of G by $\rho(\sigma) = T_\sigma$.
(a) Let χ be the character of ρ. Show that $\chi(\sigma) = |G|/|H|$ if $\sigma \in H$ and $\chi(\sigma) = 0$ if $\sigma \notin H$.
(b) Show that ρ is irreducible if and only if $H = G$.

6. (a) Let G be a finite nonabelian simple group (that is, G has no nontrivial normal subgroups). Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a two-dimensional representation of G. Show that ρ is either irreducible or trivial.
(b) It is known that the only finite subgroups of $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$ (where $\mathbb{C}^*$ denotes the multiplicative group of nonzero scalar multiples of the identity) are isomorphic to subgroups of one of the following: D_n, A_5, S_4 (where D_n is the nth dihedral group, S_n is the group of permutations of n objects, and A_n is the subgroup of even permutations). Use this fact to prove that if G is a nonabelian finite simple group with a nontrivial two-dimensional representation over $\mathbb{C}$, then $G = A_5$. (*Note:* You do not need to prove that A_5 has such a representation.)