Answers to HW 10

1. Let g be in the center of G. Then $\rho(g)$ commutes with all the matrices in $\rho(G)$. Since ρ is irreducible, Schur's Lemma implies that $\rho(g)$ is a scalar matrix $\lambda_g I$. The map $\phi : g \mapsto \lambda_g$ is a homomorphism from the center of G to $\mathbb{C}^\times$. Since ρ is injective, so is ϕ. Therefore, G is isomorphic to $\rho(G)$, which is a finite multiplicative subgroup of $\mathbb{C}^\times$, hence cyclic.

2. (a) Let $g \in G$. Then $gg_i = g_j a$ for some $a \in A$. Therefore $\rho(g)\rho(g_i)W = \rho(g_j)\rho(a)W = \rho(g_j)W$, since A is invarinat under $\rho(A)$. Therefore, $\rho(g)$ maps each g_iW into V_1, so $\rho(g)V_1 \subseteq V_1$.
(b) This follows immediately from (a) and the definition of irreducibility.
(c) If A is abelian, then W is one-dimensional. Therefore, V_1 has dimension at most d.
(d) D_n contains a cyclic subgroup of index 2. Take A to be this cyclic subgroup and $d = 2$.

3. The commutator subgroup of Q_8 is $\{\pm 1\}$. The quotient $Q_8/\{\pm 1\}$ is the Klein 4-group, so the four one-dimensional representations of the Klein 4-group give us the 4 one-dimensional representation of Q_8. Since $8 = \sum n_i^2$, there can be only one 2-dimesnional representation. The character table is

	$\{1\}$	{-1}	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

The entries for χ_5 are determined from the orthogonality of the columns.

4. (a) If M is fixed by $\tilde{\rho}$, then M commutes with $\rho(G)$. By Schur's Lemma, M is scalar. Conversely, the scalar matrics are clearly fixed by $\tilde{\rho}$. The trivial representation is given exactly by the set of fixed elements in V. Since the scalar matrices are a one-dimensional subspace, the trivial representation occurs once.
(b) We have $\mathbb{C}^2 = V_1 \oplus V_2$, where each V_i is a one-dimensional representation of G. Use a basis for $\mathbb{C}^2$ formed from a basis of V_1 and a basis of V_2. Then all of the matrices in $\rho(G)$ are diagonal. If $\rho_1 \not\simeq \rho_2$, then there exists some matrix $\rho(g) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha \neq \beta$. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is fixed by $\tilde{\rho}(g)$. Then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

This yields $b\alpha = b\beta$ and $c\alpha = c\beta$. Since $\alpha \neq \beta$, we have $b = c = 0$, so M is diagonal. The diagonal matrices are clearly fixed by conjugation by the diagonal matrices in $\rho(G)$, so the fixed elements are exactly the two dimensional space of diagonal matrices in V. Therefore, the trivial representation occurs twice. Explicitly, let W_1 be the matrices that have 0's everywhere except possibly in the upper left corner. Let V_2 be the matrices that have 0's everywhere except possibly in the lower right corner. Then $W_1 \oplus W_2$ is the sum of two trivial representations occurring in V.
(c) If ρ is the sum of two equal one-dimensional representations, then the reasoning in (b) shows that each matrix $\rho(g)$ is scalar. Therefore, $\tilde{\rho}(g)$ acts trivially on V, so V is the sum of only trivial representations. There must be 4 of them since V has dimension 4.

5. (a) The cosets $g_1H, \ldots, g_nH$ give a basis for $\mathbb{C}[G/H]$ as a vector space. If $\sigma \in G$, then multiplication by σ permutes these basis elements. The trace of T_σ is the number of basis elements that are mapped to themselves by σ. We have $\sigma g H = gH$ if and only if $\sigma H g H = gH$, which happens if and only if $\sigma H = H$. Therefore, if $\sigma \in H$, then T_σ is the identity and $\chi(\sigma) = [G : H]$, which is the dimension of the space. If $\sigma \notin H$, then the matrix for T_σ has only 0's on the diagonal, so $\chi(\sigma) = 0$.
(b) $(\chi, \chi) = |G|^{-1} \sum_{g \in G} \chi(g)\overline{\chi(g)} = |G|^{-1} \sum_{\sigma \in H} [G : H]^2 = |G|/|H|$. Therefore, χ is irreducible $\iff (\chi, \chi) = 1 \iff |G| = |H|$.

6. (a) If ρ is not irreducible, it is the sum of two one-dimensional representations. But a one-dimensional representation of G has G^c in its kernel. Since G is nonabelian and simple, $G^c = G$, so the trivial representation is the only one-dimensional representation. Therefore, if ρ is not irreducible, it is the sum of two trivial representations, which means $\rho(g) = I$ for all $g \in G$.
. (b) Consider the image of G in $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$. Since G has no nontrivial normal subgroups, the image is either trivial or isomorphic to G. If the image is trivial, then $\rho(G)$ is contained in the scalar matrices, which means that ρ is the direct sum of two one-dimensional representations, contradicting the assumption that ρ is irreducible. Therefore, the image is isomorphic to G. Among the groups D_n, A_5, S_4, only A_5 has a simple nonabelian subgroup, namely A_5. Therefore, $G = A_5$.