Homework #4

1. (a) Suppose that $0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_n \to 0$ is an exact sequence of modules, where the map $A_1 \to A_2$ is regarded as an inclusion. Show that there is an exact sequence $0 \to A_2/A_1 \to A_3 \to \cdots \to A_n \to 0$.
(b) Suppose $0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_n \to 0$ is an exact sequence of finite abelian groups. Show that $\prod_{i=1}^{n} |A_i|^{(-1)^i} = 1$ (*Remark:* abelian is not needed).
(c) Suppose $0 \to A_1 \to A_2 \to A_3 \to \cdots \to A_n \to 0$ is an exact sequence of finite-dimensional vector spaces. Show that $\sum_{i=1}^{n} (-1)^i \dim A_i = 0$.

2. Let A be an R-module and let $f : A \to A$ be a module homomorphism. If both $\mathrm{Ker} f$ and $\mathrm{Coker} f$ are finite, define $Q(A) = |\mathrm{Ker} f| / |\mathrm{Coker} f|$.
(a) Suppose $0 \to A \to B \to C \to 0$ is exact and $f : B \to B$ is such that $f(A) \subseteq A$. Show that f induces a well-defined map $C \to C$ (which we'll call f). Also show that $Q(A)Q(C) = Q(B)$ in the sense that if two of $Q(A), Q(B), Q(C)$ are defined, then so is the third and we have equality (*Hint:* Apply the Snake Lemma to two copies of $0 \to A \to B \to C \to 0$).
(b) Show that if A is finite then $Q(A) = 1$ (*Hint:* An R-module is automatically an abelian group, so Problem 1(b) applies).

3. (a) Show that if $\mathrm{Ext}^1(M, N) = 0$ for all modules N then M is projective. (*Hint:* use the long exact sequence plus an equivalent formulation of projective)
(b) Show that if $\mathrm{Ext}^1(M, N) = 0$ for all modules M then N is injective.

4. (the *Five Lemma*) Consider the following commutative diagram of modules (over some ring) with exact rows:

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & {\scriptstyle h_3}\downarrow & & {\scriptstyle h_4}\downarrow & & {\scriptstyle h_5}\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

(a) If h_2 and h_4 are surjective and h_5 is injective, prove that h_3 is surjective.
(b) If h_2 and h_4 are injective and h_1 is surjective, prove that h_3 is injective.
(c) If h_1, h_2, h_4, h_5 are isomorphisms, show that h_3 is an isomorphism.

5. Let M be a module over a ring R, and let $r \in R$. Let $h : M \to M$ be an R-module homomorphism. Let h_1 denote h restricted to rM and let h_3 denote the map on M/rM induced by h. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & rM & \longrightarrow & M & \longrightarrow & M/rM & \longrightarrow & 0 \\
 & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle h_3}\downarrow & & \\
0 & \longrightarrow & rM & \longrightarrow & M & \longrightarrow & M/rM & \longrightarrow & 0
\end{array}$$

Assume that $\mathrm{Ker}\ h$ and $\mathrm{Coker}\ h$ are finite.
(a) Show that $|\mathrm{Ker}\ h_1| \leq |\mathrm{Ker}\ h|$.
(b) Show that $|\mathrm{Coker}\ h_3| \leq |\mathrm{Coker}\ h|$.
(c) Show that $|\mathrm{Coker}\ h_1| \leq |\mathrm{Coker}\ h|$. (*Hint:* How do you get coset representatives for $rM/h_1(rM)$?)
(d) Show that $|\mathrm{Ker}\ h_3| \leq |\mathrm{Ker}\ h| \cdot |\mathrm{Coker}\ h|$.