Homework #8

1. Let ζ be a primitive 7-th root of unity. Determine how many fields F satisfy $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(\zeta)$, and for each such F determine $[F : \mathbb{Q}]$.

2. (a) Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$ and let $f(X) \in \mathbb{Q}[X]$ be the irreducible polynomial of α. The roots $\alpha_1, \dots, \alpha_m$ (including α) of $f(X)$ are called the *conjugates* of α. Let $\beta \in \mathbb{C}$ also be algebraic over $\mathbb{Q}$ and let $\beta_1, \dots, \beta_n$ be the conjugates of β. Let

$$g(X) = \prod_{i=1}^{m} \prod_{j=1}^{n} (X - \alpha_i - \beta_j).$$

Show that $g(X) \in \mathbb{Q}[X]$.
(b) Find a polynomial in $\mathbb{Q}[X]$ that has $\sqrt{2} + \sqrt{3}$ as a root.

3. Let L/K be a Galois extension with Galois group isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.
(a) Show that if $K \subseteq F \subseteq L$ then F/K is normal. (*Hint:* Figure out what property Q_8 must have, then prove it.)
(b) Let $f(X) \in K[X]$ be irreducible and have splitting field L. Show that f must have degree 8. (*Hint:* Apply (a) to $K(\alpha)$ for a root α of f, then use one of the equivalent definitions of normality.)

4. Let K be a field of characteristic 0 such that every odd degree polynomial in $K[X]$ has a root in K (for example, $K = \mathbb{R}$). Let L/K be a Galois extension and let $G = \mathrm{Gal}(L/K)$.
(a) Show that K has no nontrivial extensions of odd degree (you need a certain theorem here to make the transition from polynomials to fields; state what it is).
(b) Show that $[L : K]$ is a power of 2. (*Hint:* Look at the fixed field of the 2-Sylow subgroup of G.)
(c) Show that if $[L : K] > 2$ then there is a Galois subextension F/K with $\mathrm{Gal}(F/K)$ of order 4. (You may use the fact that in a group of order p^n for some prime p, there is a normal subgroup of order p^j for all $j \leq n$.)
(d) Let M/K be a Galois extension and let i be a square root of -1 in $\overline{K}$. Show that $M(i)/K$ is normal (any of the equivalent definitions of normality works here). Since we are in characteristic 0, this means that $M(i)/K$ is Galois.

5. Let's apply problem 4 to $K = \mathbb{R}$. Don't assume that $\mathbb{C}$ is algebraically closed.
(a) Show that $\mathbb{C}$ is the only quadratic extension of $\mathbb{R}$. (*Hint:* if $d \in \mathbb{R}$ then either d or $-d$ is a square.)
(b) Let $M/\mathbb{R}$ be a Galois extension. Show that if $[M(i) : \mathbb{R}] > 2$ then $M(i)$ contains a quadratic extension of $\mathbb{C}$. (You need problems 4(c) and 5(a) plus some very elementary group theory.)
(c) Show that $\mathbb{C}$ has no quadratic extensions. (*Hint:* If $a, b \in \mathbb{R}$, and

$$x^2 = \frac{1}{2}(\sqrt{a^2 + b^2} + a), \qquad y^2 = \frac{1}{2}(\sqrt{a^2 + b^2} - a),$$

then $x, y \in \mathbb{R}$ and (with appropriate signs of x, y) we have $(x + yi)^2 = a + bi$.)
(d) Use (b) and (c) to show that if $M/\mathbb{R}$ is a Galois extension, then $M \subseteq \mathbb{C}$.
(e) Let $f(X) \in \mathbb{C}[X]$. Show that the splitting field of $f(X)\overline{f}(X) \in \mathbb{R}[X]$ is contained in $\mathbb{C}$. Conclude that $\mathbb{C}$ is algebraically closed.

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