Solutions to Homework 9, Problems 1 and 2

1. (a) Let $L = K(\alpha, \beta, \dots)$. Then $LF = F(\alpha, \beta, = \dots)$. Since L/K is Galois, L is the splitting field of a family of separable polynomials in $F[X]$. Then LF is the splitting field of the same family, regarded as polynomials in $F[X]$.
(b) Consider the map ψ from $\mathrm{Gal}(LF/F)$ to $\mathrm{Gal}(L/K)$ given by $g \mapsto g|_L$. If $\psi(g) = 1$, then g is the identity on both F and L, therefore on LF. Therefore, ψ is injective. An element $x \in L$ is fixed by $\psi(G)$ if and only if x is in the fixed field of G, namely F. Therefore, the fixed field of $\psi(G)$ is $L \cap F$, so $\psi(G) = \mathrm{Gal}(L/(L \cap F))$.
(c) $[LF : F] = |\mathrm{Gal}(LF/F)|$, which divides $|\mathrm{Gal}(L/K)| = [L : K]$ by (b) and Lagrange's theorem.
(d) Let $K = \mathbb{Q}$, $L = \mathbb{Q}(2^{1/3})$, $F = \mathbb{Q}(\rho 2^{1/3})$, where ρ is a cube root of unity. Then $LF = \mathbb{Q}(\rho, 2^{1/3})$, which has degree 2 over F. But $[L : K] = 3$.

2. (a) This follows immediately from problem 1.
(b) Let $\psi : \mathrm{Gal}(L_1L_2/K) \to \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)$ by

$$\psi(g) = (g|_{L_1},\, g|_{L_2}).$$

Since the subgroup $\mathrm{Gal}(L_1L_2/L_2)$ maps surjectively onto $\mathrm{Gal}(L_1/K)$ by part (a), and since this subgroup maps to the identity in $\mathrm{Gal}(L_2/K)$, it follows that the image of ψ contains $\mathrm{Gal}(L_1/K) \times 1$. Similarly, the image of ψ contains $1 \times \mathrm{Gal}(L_2/K)$. Therefore, ψ is surjective. If $\psi(g) = 1$, then g is the identity on both L_1 and L_2, hence on L_1L_2. Therefore, ψ is an isomorphism. Since $\mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K) \simeq G_1 \times G_2$, we're done.