MATH 601, SAMPLE PROBLEMS

1. Let $L = \mathbb{Q}(i, 2^{1/4})$. (a) Prove that $L/\mathbb{Q}$ is Galois with $[L : \mathbb{Q}] = 8$.
(b) Determine generators and relations for $\mathrm{Gal}(L/\mathbb{Q})$.
(c) Find all fields F with $\mathbb{Q}(i) \subseteq F \subseteq L$.

2. (a) Let L/K be a finite Galois extension of fields with $\mathrm{Gal}(L/K)$ abelian. Let F be a field with $K \subseteq F \subseteq L$. Show that F/K is a Galois extension.
(b) Let $f(X) \in K[X]$ be an irreducible polynomial whose splitting field has abelian Galois group. Let α be a root of $f(X)$. Show that $K(\alpha)$ contains all roots of $f(X)$.

3. (a) Let L and L' be subfields of $\bar{\mathbb{Q}}$ such that $L' \nsubseteq L$. Show that there is an automorphism τ of $\bar{\mathbb{Q}}$ such that $\tau|_L =$id but $\tau|_{L'} \neq$id.
(b) Suppose $L/\mathbb{Q}$ is a normal extension. Let σ be an automorphism of $\bar{\mathbb{Q}}$ and let τ be an automorphism of $\bar{\mathbb{Q}}$ such that $\tau|_L =$id. Show that $(\sigma^{-1}\tau\sigma)|_L =$id.
(c) Suppose that $L/\mathbb{Q}$ is not normal. Show that there exist automorphisms τ and σ of $\bar{\mathbb{Q}}$ such that $\tau|_L =$id but $(\sigma^{-1}\tau\sigma)|_L \neq$id.

4. Let $f(x)$ be an irreducible polynomial of degree n over a field F. Let $g(x)$ be any polynomial in $F[x]$. Prove that every irreducible factor of the polynomial $f(g(x))$ has degree divisible by n.

5. Determine the splitting field F over $\mathbb{Q}$ of $X^4 + X^2 + 1$.
(b) Describe $\mathrm{Gal}(F/\mathbb{Q})$.

6. Let $L/\mathbb{Q}$ be a finite Galois extension of odd degree. Show that Ł $\subset \mathbb{R}$ (more precisely, any embedding of L into $\mathbb{C}$ has image in $\mathbb{R}$.

7. Find all irreducible polynomials of degrees 1, 2, 4 over $\mathbb{F}_2$ and prove that their product is $X^{16} - X$.