Math 620, Homework 2

1. Let K be a number field of degree n over $\mathbb{Q}$. The Minkowski bound says that each ideal class of $\mathcal{O}_K$ contains an ideal of norm at most

$$\left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|D|},$$

where D is the discriminant of K and r_2 is the number of pairs of complex embeddings $K \to \mathbb{C}$.
(a) Find all integral ideals of $\mathbb{Q}(\sqrt{5})$ of norm less than the Minkowski bound and show that all of them are principal. Conclude that the class number of $\mathbb{Q}(\sqrt{5})$ is 1.
(b) Find all integral ideals of $\mathbb{Q}(\sqrt{-13})$ of norm less than the Minkowski bound. Show that the class number of $\mathbb{Q}(\sqrt{-13})$ is 2.
(c) Every number field has the trivial ideal class and the smallest norm of an ideal in that class is 1, so the Minkowski bound must be at least 1. Use this to show that $|D_K| > 1$ if $K \neq \mathbb{Q}$.

2. Let $K/\mathbb{Q}$ be a Galois extension with abelian Galois group. Let $\alpha \in K$ be such that $|\sigma(\alpha)|^2 = r$ for some embedding $\sigma : K \to \mathbb{C}$, where $r \in \mathbb{Q}$. Show that $|\sigma(\alpha)|^2 = r$ for all embeddings $\sigma : K \to \mathbb{C}$.

3. Let $ABC \cdots XYZ$ be a closed polygon in the plane (not necessarily with 26 sides) such that every side has length 1. Suppose that the angles ABC, BCD, CDE, $\ldots$, XYZ are rational multiples of π. Show that angles YZA and ZAB are also rational multiples of π.

4. (a) Let

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$$

be an Eisenstein polynomial for the prime p, so $p \mid a_i$ for all i and $p^2 \nmid a_0$. Let α be a root of $f(X)$ and let $K = \mathbb{Q}(\alpha)$. Show that there is a prime ideal P of $\mathcal{O}_K$ dividing P such that $\alpha \in P$.
(b) Let f and P be as in (a). Show that $v_P(\alpha) = v_P(p)/n$.
(c) Use (b) to show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \geq n$, hence $= n$ (*Hint:* Relate $v_P(p)$ to a ramification index). Conclude that $f(X)$ is irreducible.