
Contents

Preface	xv
0 Introduction	1
0.1 Diophantine Equations	2
0.2 Modular Arithmetic	4
0.3 Primes and the Distribution of Primes	5
0.4 Cryptography	7
1 Divisibility	9
1.1 Divisibility	9
1.2 Euclid's Theorem	11
1.3 Euclid's Original Proof	13
1.4 The Sieve of Eratosthenes	15
1.5 The Division Algorithm	17
1.5.1 A Cryptographic Application	19
1.6 The Greatest Common Divisor	20
1.7 The Euclidean Algorithm	22
1.7.1 The Extended Euclidean Algorithm	25
1.8 Other Bases	30
1.9 Linear Diophantine Equations	32
1.10 The Postage Stamp Problem	38
1.11 Fermat and Mersenne Numbers	41
1.12 Chapter Highlights	46
1.13 Problems	46
1.13.1 Exercises	46
1.13.2 Projects	53
1.13.3 Computer Explorations	55

1.13.4	Answers to “Check Your Understanding” . .	57
2	Unique Factorization	59
2.1	Preliminary Results	59
2.2	The Fundamental Theorem of Arithmetic	61
2.3	Euclid and the Fundamental Theorem of Arithmetic	66
2.4	Chapter Highlights	67
2.5	Problems	67
2.5.1	Exercises	67
2.5.2	Projects	68
2.5.3	Answers to “Check Your Understanding” . .	70
3	Applications of Unique Factorization	71
3.1	A Puzzle	71
3.2	Irrationality Proofs	73
3.2.1	Four More Proofs That $\sqrt{2}$ Is Irrational . .	75
3.3	The Rational Root Theorem	77
3.4	Pythagorean Triples	80
3.5	Differences of Squares	86
3.6	Prime Factorization of Factorials	88
3.7	The Riemann Zeta Function	90
3.8	Chapter Highlights	96
3.9	Problems	96
3.9.1	Exercises	96
3.9.2	Projects	100
3.9.3	Computer Explorations	104
3.9.4	Answers to “Check Your Understanding” . .	105
4	Congruences	107
4.1	Definitions and Examples	107
4.2	Modular Exponentiation	115
4.3	Divisibility Tests	116
4.4	Linear Congruences	120
4.5	The Chinese Remainder Theorem	127

4.6	Fractions mod m	132
4.7	Fermat's Theorem	134
4.8	Euler's Theorem	139
4.9	Wilson's Theorem	147
4.10	Queens on a Chessboard	149
4.11	Chapter Highlights	151
4.12	Problems	151
4.12.1	Exercises	151
4.12.2	Projects	159
4.12.3	Computer Explorations	163
4.12.4	Answers to "Check Your Understanding"	164
5	Cryptographic Applications	167
5.1	Introduction	167
5.2	Shift and Affine Ciphers	170
5.3	Secret Sharing	175
5.4	RSA	177
5.5	Chapter Highlights	184
5.6	Problems	184
5.6.1	Exercises	184
5.6.2	Projects	188
5.6.3	Computer Explorations	191
5.6.4	Answers to "Check Your Understanding"	192
6	Polynomial Congruences	193
6.1	Polynomials Mod Primes	193
6.2	Solutions Modulo Prime Powers.	196
6.3	Composite Moduli	202
6.4	Chapter Highlights	203
6.5	Problems	203
6.5.1	Exercises	203
6.5.2	Projects	204
6.5.3	Computer Explorations	205

6.5.4	Answers to “Check Your Understanding” . .	206
7	Order and Primitive Roots	207
7.1	Orders of Elements	207
7.1.1	Fermat Numbers	209
7.1.2	Mersenne Numbers	211
7.2	Primitive Roots	211
7.3	Decimals	217
7.3.1	Midy’s Theorem	220
7.4	Card Shuffling	222
7.5	The Discrete Log Problem	224
7.5.1	Baby Step-Giant Step Method	226
7.5.2	The Index Calculus	228
7.6	Existence of Primitive Roots	231
7.7	Chapter Highlights	233
7.8	Problems	234
7.8.1	Exercises	234
7.8.2	Projects	238
7.8.3	Computer Explorations	239
7.8.4	Answers to “Check Your Understanding” . .	240
8	More Cryptographic Applications	241
8.1	Diffie-Hellman Key Exchange	241
8.2	Coin Flipping over the Telephone	243
8.3	Mental Poker	246
8.4	The ElGamal Public Key Cryptosystem	250
8.5	Digital Signatures	253
8.6	Chapter Highlights	255
8.7	Problems	255
8.7.1	Exercises	255
8.7.2	Projects	259
8.7.3	Computer Explorations	260
8.7.4	Answers to “Check Your Understanding” . .	260

9	Quadratic Reciprocity	263
9.1	Squares and Square Roots Mod Primes	263
9.2	Computing Square Roots Mod p	270
9.3	Quadratic Equations	272
9.4	The Jacobi Symbol	274
9.5	Proof of Quadratic Reciprocity	278
9.6	Chapter Highlights	285
9.7	Problems	286
9.7.1	Exercises	286
9.7.2	Projects	291
9.7.3	Answers to “Check Your Understanding” . .	293
10	Primality and Factorization	295
10.1	Trial Division and Fermat Factorization	295
10.2	Primality Testing	299
10.2.1	Pseudoprimes	299
10.2.2	The Pocklington-Lehmer Primality Test . .	304
10.2.3	The AKS Primality Test	307
10.2.4	Fermat Numbers	309
10.2.5	Mersenne Numbers	311
10.3	Factorization	312
10.3.1	$x^2 \equiv y^2$	312
10.3.2	Factoring Pseudoprimes and Factoring Us- ing RSA Exponents	315
10.3.3	Pollard’s $p - 1$ Method	316
10.3.4	The Quadratic Sieve	318
10.4	Coin Flipping over the Telephone	326
10.5	Chapter Highlights	328
10.6	Problems	329
10.6.1	Exercises	329
10.6.2	Projects	332
10.6.3	Computer Explorations	333
10.6.4	Answers to “Check Your Understanding” . .	334

11 Geometry of Numbers	337
11.1 Volumes and Minkowski's Theorem	337
11.2 Sums of Two Squares	342
11.2.1 Algorithm for Writing $p \equiv 1 \pmod{4}$ as a Sum of Two Squares	345
11.3 Sums of Four Squares	347
11.4 Pell's Equation	349
11.4.1 Bhāskara's Chakravala Method	353
11.5 Chapter Highlights	355
11.6 Problems	356
11.6.1 Exercises	356
11.6.2 Projects	359
11.6.3 Answers to "Check Your Understanding" . .	365
12 Arithmetic Functions	367
12.1 Perfect Numbers	367
12.2 Multiplicative Functions	371
12.3 Chapter Highlights	378
12.4 Problems	378
12.4.1 Exercises	378
12.4.2 Projects	381
12.4.3 Computer Explorations	381
12.4.4 Answers to "Check Your Understanding" . .	382
13 Continued Fractions	383
13.1 Rational Approximations; Pell's Equation	384
13.1.1 Evaluating Continued Fractions	387
13.1.2 Pell's Equation	389
13.2 Basic Theory	392
13.3 Rational Numbers	400
13.4 Periodic Continued Fractions	402
13.4.1 Purely Periodic Continued Fractions	404
13.4.2 Eventually Periodic Continued Fractions . .	409

13.5 Square Roots of Integers	411
13.6 Some Irrational Numbers	414
13.7 Chapter Highlights	420
13.8 Problems	421
13.8.1 Exercises	421
13.8.2 Projects	422
13.8.3 Computer Explorations	425
13.8.4 Answers to “Check Your Understanding” . .	425
14 Gaussian Integers	427
14.1 Complex Arithmetic	427
14.2 Gaussian Irreducibles	429
14.3 The Division Algorithm	433
14.4 Unique Factorization	436
14.5 Applications	442
14.5.1 Sums of Two Squares	442
14.5.2 Pythagorean Triples	445
14.5.3 $y^2 = x^3 - 1$	447
14.6 Chapter Highlights	448
14.7 Problems	449
14.7.1 Exercises	449
14.7.2 Projects	450
14.7.3 Computer Explorations	450
14.7.4 Answers to “Check Your Understanding” . .	450
15 Algebraic Integers	453
15.1 Quadratic Fields and Algebraic Integers	453
15.2 Units	458
15.3 $\mathbb{Z}[\sqrt{-2}]$	462
15.4 $\mathbb{Z}[\sqrt{3}]$	466
15.4.1 The Lucas-Lehmer Test	469
15.5 Non-unique Factorization	472
15.6 Chapter Highlights	474

15.7 Problems	475
15.7.1 Exercises	475
15.7.2 Projects	476
15.7.3 Answers to “Check Your Understanding” . .	478
16 Analytic Methods	479
16.1 $\sum 1/p$ Diverges	479
16.2 Bertrand’s Postulate	485
16.3 Chebyshev’s Approximate Prime Number Theorem	493
16.4 Chapter Highlights	499
16.5 Problems	499
16.5.1 Exercises	499
16.5.2 Projects	500
16.5.3 Computer Explorations	501
17 Epilogue: Fermat’s Last Theorem	503
17.1 Introduction	503
17.2 Elliptic Curves	506
17.3 Modularity	510
A Supplementary Topics	513
A.1 Geometric Series	513
A.2 Mathematical Induction	515
A.3 Pascal’s Triangle and the Binomial Theorem	521
A.4 Fibonacci Numbers	526
A.5 Problems	530
A.5.1 Exercises	530
A.5.2 Answers to “Check Your Understanding” . .	532
B Answers and Hints for Odd-Numbered Exercises	535
Index	549