# Mapl 612: Homework Assignment 1

due Thursday, 8 March 2001

Let $U(x,t)$ satisfy

$$\begin{aligned}
\partial_t U = \kappa \partial_{xx} U \qquad & \text{for } x \in (0,\pi),\ t > 0\,;\\
U(0,t) = U_L\,, \quad \partial_x U(\pi,t) = 0 \qquad & \text{for } t > 0\,;\\
U(x,0) = U^{in}(x) \qquad & \text{for } x \in (0,\pi)\,.
\end{aligned}$$

Here $\kappa > 0$ is a constant diffusion coefficient, $U_L > 0$ is a boundary value, and $U^{in} \geq 0$ is initial data.

For every $J \in \{1, 2, \cdots\}$ set $\delta x = \pi/J$ and define $\mathbf{u}^{in} \in \mathbb{R}^J$ by

$$\mathbf{u}^{in} = \begin{pmatrix} U^{in}(\delta x) \\ U^{in}(2\delta x) \\ \vdots \\ U^{in}(J\delta x) \end{pmatrix} .$$

For every $\theta \in [0,1]$ let $\mathbf{u}^n \in \mathbb{R}^J$ satisfy the weighted difference scheme

$$\begin{gathered}
\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} = \mathbf{A}\big(\theta \mathbf{u}^{n+1} + (1-\theta)\mathbf{u}^n\big) + \mathbf{b} \quad \text{for } n \in \{0, 1, \cdots\}\,,\\
\mathbf{u}^0 = \mathbf{u}^{in}\,,
\end{gathered}$$

where $\mathbf{A} \in \mathbb{R}^{J\times J}$ is defined by

$$\mathbf{A} = \frac{\kappa}{(\delta x)^2} \begin{pmatrix}
-2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & -2 & 1 & \ddots & & \vdots \\
0 & 1 & -2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 0 \\
\vdots & & \ddots & 1 & -2 & 1 \\
0 & \cdots & \cdots & 0 & 2 & -2
\end{pmatrix},$$

and $\mathbf{b} \in \mathbb{R}^J$ is defined by

$$\mathbf{b} = \frac{\kappa}{(\delta x)^2} \begin{pmatrix} U_L \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Set $r = \kappa \delta t/(\delta x)^2$.

(a) Show that this scheme is consistent. Compute and bound the truncation error as a function of $\theta$ and $r$. You may assume that $U$ has as many spatial derivatives as you need formally. Your bound may be expressed in terms of

$$M_k(t) = \max\left\{ \left|\partial_x^k U(x,t)\right| \, : \, x \in [0,\pi] \right\}$$

for some $k$.

(b) Analyze the stability of the growth matrix $\mathbf{G}$ as a function of $\theta$ and $r$.

(c) Assume the functions $M_k(t)$ that you used in (a) can be bounded above independent of $t$ by constants $M_k^{in}$ that depend only on $U^{in}$. Combine the results of (a) and (b) to give a bound on the discrete error whenever the scheme is stable.

(d) Set $\kappa = 1$ and $U_L = 1$. Compute the numerical solution at $t = 2$ for

| $J$ | $\theta$ | $r$ |
|---|---|---|
| $10, 20$ | $0$ | $.5$ |
| $10, 20$ | $.25$ | $.5$ |
| $10, 20$ | $.5$ | $.5, 2.5$ |
| $10$ | $1$ | $.5, 2.5$ |
| $20$ | $1$ | $2.5, 50$ |

with

$$U^{in}(x) = 1 - \frac{x}{\pi}\,, \qquad U^{in}(x) = \begin{cases} 1 & \text{for } 0 \le x < \pi/2 \\ .5 & \text{for } x = \pi/2 \\ 0 & \text{for } \pi/2 < x \le \pi \end{cases} .$$

Make comments relating your results to (a-c).