Mapl 612: Homework Assignment 2

due Thursday, 29 March 2001

Let $U(x,t)$ satisfy

$$\begin{aligned} \eta(x)\partial_t U &= \partial_x[a(x)\partial_x U] && \text{for } x \in (0,1),\, t > 0\,; \\ U(0,t) &= 0 && \text{for } t > 0\,; \\ -a(1)\partial_x U(1,t) &= b_R(U(1,t) - U_R) && \text{for } t > 0\,; \\ U(x,0) &= U^{in}(x) && \text{for } x \in (0,1)\,. \end{aligned} \tag{PDE}$$

Here $\eta(x) > 0$ is a variable heat capacity, $a(x) > 0$ is a variable thermal conductivity, $b_R > 0$ is a boundary cooling coefficient, $U_R > 0$ is a boundary value, and $U^{in} \geq 0$ is initial data.

1) Show that PDE formally has the spatial weak formulation:

$$\int_0^1 \eta(x)\phi(x)\partial_t U(x,t)\,dx = -\int_0^1 a(x)\partial_x\phi(x)\partial_x U(x,t)\,dx - b_R\phi(1)\big(U(1,t) - U_R\big)$$

for every $\phi \in C^1([0,1])$ such that $\phi(0) = 0$.

2) Divide $(0,1)$ into J subintervals (cells, zones, ...) with endpoints

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{J+\frac{1}{2}} = 1\,.$$

The width of the j^{th} cell is then $\delta_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. Suppose that $\eta(x)$ and $a(x)$ are constant over each cell. Let η_j and a_j denote their respective values over the j^{th} cell.

Use the weak formulation of problem (1) to derive a semidiscrete spatial approximation. Let u_j approximate the spatial average of U over the j^{th} cell. Assume that U is spatially approximated by functions that are quadratic over each cell, have average value u_j over the j^{th} cell, and are continuous at each node. For the spatial-derivative term assume that ϕ has the same form, while for the time-derivative term assume that ϕ is constant over each cell. The time-derivative term is therefore approximated by the sum

$$\sum_{j=1}^{J} \phi_j \frac{du_j}{dt}\eta_j\delta_j\,.$$

3) Put the scheme you derive into the form

$$\mathbf{M}\frac{d\mathbf{u}}{dt} + \mathbf{A}\mathbf{u} = \mathbf{b}\,, \qquad \mathbf{u}(0) = \mathbf{u}^{in}\,,$$

where $\mathbf{u}(t) \in \mathbb{R}^J$ while $\mathbf{M}$, $\mathbf{A} \in \mathbb{R}^{J\times J}$ and

$$\mathbf{u}^{in} = \begin{pmatrix} u_1^{in} \\ u_2^{in} \\ \vdots \\ u_J^{in} \end{pmatrix}, \qquad u_j^{in} = \frac{1}{\delta_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U^{in}(x)\,dx\,.$$

Identify $\mathbf{M}$, $\mathbf{A}$, and $\mathbf{b}$. Give an inner product over $\mathbb{R}^J$ for which $\mathbf{M}$ and $\mathbf{A}$ are symmetric and postive definite.

4) Put the scheme you derive into the conservative form

$$\partial_t \rho_j + \frac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\delta_j} = 0 \quad \text{for } j = 1, 2, \cdots, J\,,$$

where ρ_j approximates the spatial average of ηU over the j^{th} cell while $f_{j+\frac{1}{2}}$ approximates the flux $-a(x)\partial_x U$ at the node $x_{j+\frac{1}{2}}$.

5) Let η and a be given by

$$\eta(x) = \begin{cases} .5 & \text{for } x \in [0, .5)\,, \\ 1. & \text{for } x \in (.5, 1]\,, \end{cases} \qquad a(x) = \begin{cases} 4. & \text{for } x \in [0, .5)\,, \\ 1. & \text{for } x \in (.5, 1]\,. \end{cases}$$

Let $U_R = 1$, $b_R = 3$, and $U^{in}(x) = 1$. For each $J = \{10, 20, 40, 80\}$ consider two spatial grids: uniform ($\delta_j = \frac{1}{J}$) and alternating ($\delta_j = \frac{2-(-1)^j}{2J}$). Use the implicit Euler scheme for your temporal differencing with $\delta t = \frac{1}{J}$.

(a) Plot the numerical solution at $t = .2$ and $t = 2$.

(b) Compute the discrete error at $t = .2$ and $t = 2$.

(c) Plot $\log(\|\text{error}\|_p)$ vs. $\frac{1}{J}$ to estimate the rate of convergence for $p = 1, 2, \infty$ (if any).

Make comments about your results to (a-c).