Mapl 612: Homework Assignment 3

due Tuesday, 15 May 2001

Let G be a symmetric positive definite $N\times N$-matrix. Let A be an $N\times N$-matrix such that $(GA)^T = GA$. Let $U(x,t)$ take values in $\mathbb{R}^N$ and formally satisfy

$$\partial_t U + A\partial_x U = 0\,, \qquad U(x+2\pi,t) = U(x,t)\,, \qquad U(x,0) = U^{in}(x)\,, \tag{PDE}$$

where $U^{in}(x+2\pi) = U^{in}(x)$. Let D be an $N\times N$-matrix that commutes with A (i.e. $DA = AD$) and is symmetric nonnegative definite with respect to G (i.e. $(GA)^T = GA \geq 0$). Consider a uniform spatial grid of K uniform subintervals over $[0,2\pi]$ and the three-point viscosity scheme

$$u_k^{n+1} = \frac{1}{2}\left(\frac{\delta_t}{\delta_x^2}D + \frac{\delta_t}{\delta_x}A\right)u_{k-1}^n + \left(I - \frac{\delta_t}{\delta_x^2}D\right)u_k^n + \frac{1}{2}\left(\frac{\delta_t}{\delta_x^2}D - \frac{\delta_t}{\delta_x}A\right)u_{k+1}^n\,. \tag{VS}$$

Here $\delta_x = 2\pi/K$ is the grid size and δ_t is the time step.

It was shown in class that if $\hat{u}$ is a joint eigenvector of both A and D such that

$$\frac{\delta_t}{\delta_x}A\hat{u} = \lambda\hat{u}\,, \qquad \frac{\delta_t}{\delta_x^2}D\hat{u} = \mu\hat{u}\,,$$

then the scheme (VS) has solutions of the form $u = e^{i(\xi x - \omega t)}\hat{u}$, where ω is given by the dispersion relation

$$e^{-i\omega\delta_t} = 1 - 2\mu\sin^2(\tfrac{1}{2}\xi\delta_x) - i\lambda\sin(\xi\delta_x)\,. \tag{D}$$

1) Show that solutions of (PDE) formally satisfy the local conservation law

$$\partial_t\big(U^T G U\big) + \partial_x\big(U^T G A U\big) = 0\,.$$

Show that they also formally satisfy the global conservation law

$$\int_0^{2\pi}\big(U(x,t)^T G U(x,t)\big)\,dx = \int_0^{2\pi}\big(U^{in}(x)^T G U^{in}(x)\big)\,dx\,.$$

2) Use the dispersion relation (D) to show that the scheme (VS) is stable in the sense that

$$\sum_{k=1}^{K}\big((u_k^{n+1})^T G u_k^{n+1}\big)\delta_x \leq \sum_{k=1}^{K}\big((u_k^n)^T G u_k^n\big)\delta_x$$

if and only if

$$\left(\frac{\delta_t}{\delta_x}A\right)^2 \leq \frac{\delta_t}{\delta_x^2}D \leq I\,.$$

3) Use the dispersion relation (D) to compute the modified equation for the scheme (VS) that captures the leading dissipative and dispersive contributions to the error for resolved wavelengths (i.e. for $|\xi \delta_x| \ll 1$). Consider two cases:

(a) $D = \delta_t A^2$ (Lax-Wendroff),

(b) $D > \delta_t A^2$.

For each case show how the evolution of the quantity $(U^T G U)$, which is locally and globally conserved when U satisfies (PDE), differs when U satisfies the modified equation. Contrast the two cases.

4) Consider the special $N = 2$ case of (PDE) given by

$$G = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad U^{in}(x) = \begin{pmatrix} w(x) \\ w(x) \end{pmatrix},$$

where w is the 2π-periodic function defined over $[0, 2\pi]$ by

$$w(x) = \max\{0\,,\, \pi - 4|x - \pi|\}\,.$$

Solve this numerically over the interval $[0, 2\pi]$ for $K = 40$ and $K = 160$ with $\delta_t = \delta_x$ and $\delta_t = \delta_x/3$ using

(a) $D = \delta_t A^2$ (Lax-Wendroff),

(b) $D = \delta_x |A|$ (upwind),

(c) $D = \delta_x r_{sp}(A) I$ (scalar viscosity),

(d) $D = \delta_x^2/\delta_t I$ (Lax-Friedrichs),

There at first may appear to be 16 cases here. However, because of the special form of A and a special value of δ_t, there are only eight. For each distinct case make one graph that plots u and v at times $t = 2\pi$ and $t = 4\pi$ vs x. (So that four functions are plotted on each graph.) Explain what you see.

5) Consider the spatially-periodic scalar conservation law

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) = 0\,, \qquad u(x+1, t) = u(x, t)\,, \qquad u(x, 0) = \sin(2\pi x)\,.$$

Solve this numerically over the interval $[0, 1]$ for $K = 40$ and $K = 160$ with $\delta_t = \delta_x$ and $\delta_t = \delta_x/2$ using

(a) the Godunov scheme,

(b) the van-Leer scheme (piecewise linear) with the van-Leer limiter (harmonic mean).

For each case make one graph that plots u at times $t = 0$, $t = .5$, $t = 1$, and $t = 2$ vs x. (So that four functions are plotted on each graph.) Explain what you see.