Numerical Analysis I: AMSC/CMSC 666
Homework 1, due Friday, 15 October

1. Let $f(x) = e^x$. Give the Newton form of the polynomial that interpolates the values of f at $x = -1$, 0, and 1. Give a uniform bound on the error of this approximation over $[-1, 1]$.

2. Let $f(x) = e^x$. Give the quadratic Chebyschev interpolation of f over $[-1, 1]$. Give a uniform bound on the error of this approximation over $[-1, 1]$.

3. Define the inner product $(\,\cdot\,|\,\cdot\,)$ on $C([-1, 1])$ by
$$(\, f \,|\, g \,) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)\, g(x)}{\sqrt{1 - x^2}} \, \mathrm{d}x \,.$$
Show that the Chebyschev polynomials T_n are mutually orthogonal with respect to this inner product.

4. Stoer and Bulirsch (third edition), page 138, problem 16.

5. Partition the interval $[x_L, x_R]$ into n subintervals as
$$x_L = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = x_R \,.$$
Prove that the linear spline minimizes the integral
$$\int_{x_L}^{x_R} |Y'(x)|^2 \, \mathrm{d}x \,,$$
subject to the constraints $Y(x_i) = y_i$ for $i = 0, \cdots, n$ over the class of functions Y that are continuous over $[x_L, x_R]$ and are smooth over the subinterval (x_{i-1}, x_i) for every $i = 1, \cdots, n$.

6. Partition the interval $[x_L, x_R]$ into n subintervals as
$$x_L = x_0 < x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} < x_n = x_R \,.$$
Find the quadratic spline Y that interpolates the data
$$\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} Y(x) \, \mathrm{d}x = y_i \quad \text{for } i = 1, \cdots, n \,,$$
and that satifies $Y'(x_L) = \dot{y}_L$ and $Y'(x_R) = \dot{y}_R$.

Project. Consider the function $f(x) = 1/(1 + e^x)$ over $[-5, 5]$. Make graphs that compare the following six interpolations of f:

- the Lagrange interpolations through $(x_i, f(x_i))$ where $x_i = \pm 1, \pm 3, \pm 5$, and where $x_i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.
- the Chebyshev interpolations of degrees 5 and 10 over $[-5, 5]$.
- the cubic splines through $(x_i, f(x_i))$ where $x_i = \pm 1, \pm 3, \pm 5$, and where $x_i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$.

Explain the differences you see. You might find it helpful to program the divided differences algorithm (Stoer and Bulirsch section 2.1.3) to generate the polynomials asked for in the first two items above.