Vector and Matrix Norms
David Levermore
draft October 11, 2004

Vector Norms and Distance. A linear space (which is also called a vector space) is often endowed with a *vector norm* — a real-valued function that measures the size (also referred to as length or magnitude) of its vectors. The norm of a vector x is denoted $\|x\|$. This notation indicates that it is an extension of the idea of the absolute value of a number. A vector norm satisfies the following properties for any vectors x, y, and scalar α:

(1a) $$\|x\| \geq 0\,,$$
(1b) $$\|x\| = 0 \quad \text{if and only if} \quad x = 0\,,$$
(1c) $$\|x + y\| \leq \|x\| + \|y\|\,,$$
(1d) $$\|\alpha x\| = |\alpha|\,\|x\|\,.$$

In words, the first property states that no vector has negative length, the second that only the zero vector has zero length, the third that the length of a sum is no greater than the sum of the lengths (the so-called *triangle inequality*), and the fourth that the length of a multiple is the magnitude of the multiple times the length. Any real-valued function satisfying these properties can be a vector norm.

Given any vector norm $\|\cdot\|$, the distance between any two vectors x and y is defined to be $\|y - x\|$. In other words, the distance between two vectors is the length of their difference. A sequence of vectors $x^{(n)}$ is said to converge to the vector x when the sequence of nonnegative numbers $\|x^{(n)} - x\|$ converges to zero — in other words, when the distance between $x^{(n)}$ and x vanishes as n tends to infinity.

When the linear space is $\mathbf{R}^N$ with real scalars the most common choices for vector norms have the form

(2)
$$\begin{aligned}
\|x\|_\infty &= \max_{1\leq i\leq N} \Big\{ |x_i| \Big\}\,, \\
\|x\|_2 &= \left(\sum_{i=1}^{N} |x_i|^2 w_i \right)^{\frac{1}{2}}\,, \\
\|x\|_1 &= \sum_{i=1}^{N} |x_i| w_i\,,
\end{aligned}$$

where $w = (w_1, w_2, \cdots, w_N)$ is a given vector of positive weights. The first of these is the norm introduced in the so-called uniform norm. It arises naturally when studying the error of numerical methods. The second corresponds to the notion of Euclidian length that you may recall from when you first learned about vectors. Here it is given in a general N dimensional case. The third arises naturally in systems in which the sum of the variables x_i is conserved with respect to the weights w_i. For example, when the $x_i w_i$ represent the mass or energy of components of a system in which the total mass or energy is conserved.

There are many other choices for vector norms over $\mathbf{R}^N$. For example, the norms given in (2) are part of a larger family of norms which for any $1 \leq p < \infty$ includes

$$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p w_i \right)^{\frac{1}{p}}\,.$$

The choice of norm to be used in a given application depends on the physical meaning of x in that application. For example, in problems where the x is a vector of velocities (say in a fluid dynamics simulation) then $\|\cdot\|_2$ may be the most natural norm because half its square is the kinetic energy.

We will not prove that the norms given in (2) do indeed satisfy the properties given in (1). This is really not too difficult to do, and is usually covered in appropriate mathematics courses. I expect you to know how to do this. However, from a practical standpoint, knowing how to prove the properties given in (1) is not as important as knowing how to use them. That is because once those properties have been established, they are pretty much all you need. Mathematicians take the properties given in (1) as the definition of a norm and study their implications. That way once you know something is a norm, you know a whole bunch of things you can do with it that do not depend on the details of its formula.

When $1 \leq p \leq q < \infty$ the above norms are related by the inequalities

$$C_{min} \|x\|_\infty \leq C_{min}^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \leq \|x\|_p \leq C_{sum}^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \leq C_{sum} \|x\|_\infty \,,$$

where the constants C_{min} and C_{sum} are given by

$$C_{min} = \min_{1 \leq i \leq N} \{w_i\} \,, \qquad C_{sum} = \sum_{i=1}^{N} w_i \,.$$

For example, when $w_i = 1$ for every i one has $C_{min} = 1$ and $C_{sum} = N$. These inequalities show that when a sequence $x^{(n)}$ converges to x in one of these norms, it converges to x in all of these norms.

These norms are naturally related to the inner product

$$(x|y) = \sum_{i=1,N} x_i y_i w_i \,.$$

Indeed, one has that $\|x\|_2 = \sqrt{(x|x)}$ and that

$$|(x|y)| \leq \|x\|_p \|y\|_{p^*} \quad \text{where } \frac{1}{p} + \frac{1}{p^*} = 1 \,.$$

Here we understand that $p^* = \infty$ when $p = 1$. This is called the Hölder inequality. The special case $p = 2$ is called the Cauchy-Schwarz inequality.

Matrix Norms. A matrix norm is a real-valued function that measures the size of matrices by how they effect the size of any vector they multiply. For a given vector norm, the associated norm of a square matrix A is denoted $\|A\|$ and defined by

(3)
$$\|A\| = \max \left\{ \frac{\|Ax\|}{\|x\|} \; : \; x \neq 0 \right\} .$$

This definition states that $\|A\|$ is the lagrest factor by which the norm of a vector will be changed upon multiplication by the matrix A. It is therefore clear that for any vector x one has

(4)
$$\|Ax\| \leq \|A\| \, \|x\| \,.$$

The fact that similar notation is used to denote both vector and matrix norms may be confusing at first. The way to keep them straight is by looking at the object inside the $\| \cdot \|$: if that object is a vector like x or Ax then you have a vector norm; if it is a matrix like A then you have a matrix norm.

The matrix norms associated to the vector norms $\| \cdot \|_\infty$, $\| \cdot \|_2$, and $\| \cdot \|_1$ are:

(5)
$$\begin{aligned} \|A\|_\infty &= \max_{1 \leq i \leq N} \left\{ \sum_{j=1}^{N} |a_{ij}| w_j \right\} , \\ \|A\|_2 &= \max \left\{ \lambda^{\frac{1}{2}} \; : \; \lambda \text{ is an eigenvalue of } A^* A \right\} , \\ \|A\|_1 &= \max_{1 \leq j \leq N} \left\{ \sum_{i=1}^{n} |a_{ij}| w_i \right\} . \end{aligned}$$

Here A^* is the adjoint of A with respect to the inner product $(\cdot \,|\, \cdot)$. It is given by

$$A^* = W^{-1} A^T W \,,$$

where W is the diagonal matrix with w on the diagonal and A^T is the transpose of A.

The first and third of these matrix norms are always easy to compute, while the second gets more and more complicated as N increases. The second can however be simply bounded above by the first an third as

$$\|A\|_2 \leq \sqrt{\|A\|_\infty \|A\|_1} \,.$$

In practice this upper bound is quite useful.

As an example, consider A given by

$$A = \begin{pmatrix} 10 & 9 \\ 1 & 1 \end{pmatrix} .$$

One can easily see that

$$\|A\|_\infty = 19 \,, \qquad \|A\|_1 = 11 \,,$$

while

$$\|A\|_2 \leq \sqrt{19 \cdot 11} = \sqrt{209} \leq 14.5 \,.$$

The exact value of $\|A\|_2$ is the square root of the largest eigenvalue of

$$A^* A = \begin{pmatrix} 10 & 1 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 10 & 9 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 101 & 91 \\ 91 & 82 \end{pmatrix} .$$

This has a value that is a bit less that 13.6, so the upper bound is not too bad.

We will not prove that the norms given in (5) are indeed the matrix norms associated by definition (3) to the vector norms given in (2). This is a bit tricky, and is usually covered in appropriate mathematics courses. At this stage you just need to know how to use these matrix norms. Specifically, you should know that any matrix norm given by (3) satisfies the following properties for any matrices A, B, scalar α, and vector x:

(6a) $\|A\| \geq 0 \,,$

(6b) $\|A\| = 0 \quad \text{if and only if} \quad A = 0 \,,$

(6c) $\|A + B\| \leq \|A\| + \|B\| \,,$

(6d) $\|\alpha A\| = |\alpha| \, \|A\| \,,$

(6e) $\|AB\| \leq \|A\| \, \|B\| \,,$

(6f) $\|I\| = 1 \,,$

(6g) $\|Ax\| \leq \|A\| \, \|x\| \,.$

Here I denotes the identity matrix. The first four of these (6a-d) just reflect the fact that a matrix norm given by (3) is indeed a norm. The distance between two matrices A and B is then given by $\|B - A\|$. The last three (6e-g) also follow from (3). Indeed, (6f) is obvious, while (6g) is just (4). We will not prove that any matrix norm given by (3) does indeed satisfy all the properties given in (6). This is really not too difficult to do, and is usually covered in appropriate mathematics courses.

Application to Fixed-Point Interation Algorithms. You have already seen how vector and matrix norms are useful tools to help you understand the stability of solving a linear system $Ax = b$. A key quantity that arose in that analysis was the condition number of the invertible matrix A, which was defined by

$$\mathrm{cond}(A) = \|A\| \, \|A^{-1}\| \,.$$

Notice that in applications where the entries of A all have the same dimensional units, those of A^{-1} will all have the reciprocal dimensional units. You can see from definition (3) that the norms of such matrices have the same dimensional units as the matrices. It is therefore clear that cond(A) is nondimensional. This means its value is unchanged by converting the problem to different units. It is therefore an intrinsic measure of the mathmatical difficulty of solving the problem.