Take Home Exam: AMSC/CMSC 666
due 5pm, Wednesday, 15 December

(1) Let $Q_\Delta(f)$ denote quadrature over an interval by the trapezoidal rule with uniform subintervals of length Δ. Use the Euler-Maclaurin formula to extrapolate Q_Δ, $Q_{2\Delta}$, $Q_{3\Delta}$, and $Q_{6\Delta}$ to obtain an eighth order accurate quadrature.

(2) Derive the one-, two-, three-, and four-point Gaussian quadrature formulas such that

$$\int_{-1}^{1} f(x)x^2 \, \mathrm{d}x = \sum_{j=1}^{n} f(x_j)\, w_j \,.$$

Given bounds on the error of these formulas.

(3) We wish to solve $Ax = b$ iteratively where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} .$$

Show that the Jacobi Method converges while the Gauss-Seidel Method does not. For what values of the parameter ω does the SOR method converge?

(4) Let $A \in \mathbb{R}^{N\times N}$ be self-adjoint and positive definite with respect to a distinguished real inner product $(\,\cdot\,|\,\cdot\,)$ over $\mathbb{R}^N$. Let $b \in \mathbb{R}^N$. Define

$$f(y) = (y\,|\,Ay) - 2(b\,|\,y) \quad \text{for every } y \in \mathbb{R}^N .$$

Consider the steepest descent method to solve $Ax = b$:

choose an initial iterate $x^{(0)} \in \mathbb{R}^N$;

compute the initial residual $r^{(0)} = b - Ax^{(0)}$;

$$\alpha_n = \frac{\big(r^{(n)}\,|\,r^{(n)}\big)}{\big(r^{(n)}\,|\,Ar^{(n)}\big)} \,;$$

$$x^{(n+1)} = x^{(n)} + \alpha_n r^{(n)} \,;$$

$$r^{(n+1)} = r^{(n)} - \alpha_n A r^{(n)} \,.$$

Let $e^{(n)} = x^{(n)} - x$ be the error of the n^{th} iterate.

(a) Let κ be the condition number of A. Prove that for every nonzero $y \in \mathbb{R}^N$ one has

$$1 \le \frac{(y\,|\,Ay)(y\,|\,A^{-1}y)}{(y\,|\,y)^2} \le \frac{(\kappa+1)^2}{4\kappa} \,.$$

Hints: Lagrange multipliers and diagonalize.

(b) Prove that

$$\frac{\left\|e^{(n+1)}\right\|_A^2}{\left\|e^{(n)}\right\|_A^2} = 1 - \frac{\left(r^{(n)} \mid r^{(n)}\right)}{\left(r^{(n)} \mid Ar^{(n)}\right)} \frac{\left(r^{(n)} \mid r^{(n)}\right)}{\left(r^{(n)} \mid A^{-1}r^{(n)}\right)} \,,$$

where $\| \cdot \|_A$ denotes the A-norm.

(c) Use the above facts to derive a bound on $\left\|e^{(n)}\right\|_A$ in terms of κ and $\left\|e^{(0)}\right\|_A$. Compare the result with the similar estimate derived in class for the conjugate gradient method.

(5) Let A be the symmetric tridiagonal real matrix

$$A = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix} . \tag{1}$$

Show that A is irreducible if and only if every b_m is nonzero.

(6) Let A be an irreducible symmetric tridiagonal real matrix of the form (1). Let $\{p_m(x)\}_{m=0}^{n+1}$ be the sequence of polynomials generated by

$$p_0(x) = 1 \,, \qquad p_1(x) = (x - a_0) \,,$$
$$p_{m+1}(x) = (x - a_m)p_m(x) - b_m^2 p_{m-1}(x) \quad \text{for } m = 1, \cdots, n \,.$$

Let $\pi_0 = 1$, and $\pi_m = b_m \pi_{m-1}$ for every $m = 1, \cdots, n$. Let $q_m(x) = p_m(x)/\pi_m$ for every $m = 0, \cdots, n$.

(a) Show that $p_{n+1}(x)$ has $n+1$ simple roots $\{x_m\}_{m=0}^{n+1}$.

(b) Show that $V^{-1}AV$ is diagonal where

$$V = \begin{pmatrix} q_0(x_0) & q_0(x_1) & q_0(x_2) & \cdots & q_0(x_n) \\ q_1(x_0) & q_1(x_1) & q_1(x_2) & \cdots & q_1(x_n) \\ q_2(x_0) & q_2(x_1) & q_2(x_2) & \cdots & q_2(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n(x_0) & q_n(x_1) & q_n(x_2) & \cdots & q_n(x_n) \end{pmatrix} .$$

(7) Given any self-adjoint matrix $A \in \mathbb{R}^{N\times N}$ and any unit vector $u \in \mathbb{R}^N$, use the Lanczos algorithm to construct an orthogonal matrix Q such that the first column of Q is u and that $Q^T AQ$ is tridiagonal.

(8) Recall that $A \in \mathbb{C}^{N\times N}$ is called *normal* whenever $A^*A = AA^*$. Show that A is normal and invertible if and only if there exists a unitary matrix U and a self-adjoint, positive definite matrix P such that $A = UP = PU$.

(9) Let $A \in \mathbb{R}^{N \times N}$ be normal and invertible. Let $\{A_n\}_{n=0}^{\infty}$ be the sequence of $N \times N$ matricies constructed recursively by the QR-Method: $A_0 = A$, $A_n = Q_n R_n$, and $A_{n+1} = R_n Q_n$, where every Q_n is orthogonal and every R_n is upper triangular with positive diagonal entries. Show that every A_n is normal. (Hint: The result of the previous problem might be helpful.)

(10) Let $H_0 \in \mathbb{R}^{N \times N}$ and $H(t)$ satisfy the isospectral flow initial value problem

$$\frac{\mathrm{d}H}{\mathrm{d}t} = JH - HJ \,, \qquad H(0) = H_0 \,,$$

where $J(t) \in \mathbb{R}^{N \times N}$ such that $J(t)^T = -J(t)$ for every $t \in \mathbb{R}$. Show that if H_0 is normal then so is $H(t)$ for every $t \in \mathbb{R}$.