Second In-Class Exam Solutions: MATH 410
Wednesday, 9 November 2005

1. [30] State whether each of the following statements is true or false. Give a proof when true and a counterexample when false.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is monotonic and one-to-one then it is also continuous.

Solution: This statement is *false*. There are many counterexamples. The simplest ones have a single jump discontinuity somewhere. For example, consider the function f defined by

$$f(x) = \begin{cases} x & \text{for } x < 0\,, \\ x + 1 & \text{for } x \geq 0\,. \end{cases}$$

This function is clearly increasing and one-to-one, but has a jump discontinuity at $x = 0$ because

$$\lim_{x \to 0^-} f(x) = 0 \neq 1 = \lim_{x \to 0^+} f(x)\,.$$

$\square$

(b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then it is continuous.

Solution: This statement is *true*. Let $c \in \mathbb{R}$ be arbitrary. Because f is differentiable at c we know that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)\,.$$

Because for every $x \in \mathbb{R}$ such that $x \neq c$ one has

$$f(x) = f(c) + \frac{f(x) - f(c)}{x - c}\,(x - c)\,,$$

it follows from the algebraic properties of limits that

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c) \\ &= f(c) + f'(c) \cdot 0 = f(c)\,. \end{aligned}$$

Hence, f is continuous at c. But $c \in \mathbb{R}$ was arbitrary. Hence, f is continuous over $\mathbb{R}$. $\square$

Remark: The facts

$$\lim_{x \to c} f(c) = f(c)\,, \quad \text{and} \quad \lim_{x \to c}(x - c) = 0\,,$$

were used above without fanfare. You do not have to give proofs of such elementary facts unless you are explicitly asked to do so.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution: This statement is *false*. The example we had discussed in class was

$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

This function is clearly differentiable at every $x \neq 0$ with

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \quad \text{for } x \neq 0\,.$$

Moreover, it is differentiable at $x = 0$ with

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \\ &= \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0\,. \end{aligned}$$

Hence, f is differentiable over $\mathbb{R}$. However, because

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist}\,,$$

while

$$\lim_{x \to 0} 2x \cos\left(\frac{1}{x}\right) = 0\,,$$

it follows that

$$\lim_{x \to 0} f'(x) \quad \text{does not exist}\,.$$

Therefore f' is not continuous at $x = 0$. □

2. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{N}$. Suppose the following equation has at most n solutions:

$$f'(x) = 0\,, \qquad x \in \mathbb{R}\,.$$

Show the following equation has at most $n + 1$ solutions:

$$f(x) = 0\,, \qquad x \in \mathbb{R}\,.$$

Solution: Suppose that the equation $f'(x) = 0$ has at most n solutions while the equation $f(x) = 0$ has $n + 2$ solutions $\{x_i\}_{i=0}^{n+1}$. Without loss of generality we can assume these points are labeled so that

$$-\infty < x_0 < x_1 < \cdots < x_n < x_{n+1} < \infty\,.$$

Then for each $i = 1, \cdots, n+1$ one knows that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, \cdots, n+1$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the $n + 1$ intervals (x_{i-1}, x_i) are disjoint, the points p_i are distinct. The equation $f'(x) = 0$ therefore has at least $n + 1$ solutions, which contradicts our starting supposition. $\square$

Alternative Solution: Suppose $f'(x) = 0$ has exactly m solutions $\{c_i\}_{i=1}^{m}$, where $m \leq n$. Without loss of generality we can assume these m critical points are labeled so that

$$-\infty < c_1 < c_2 < \cdots < c_{m-1} < c_m < \infty\,.$$

By the Dichotomy Theorem f' must be either negative or positive over each of the $m + 1$ disjoint intervals

$$(-\infty, c_1)\,, \quad (c_1, c_2)\,, \quad \cdots \quad (c_{m-1}, c_m)\,, \quad (c_m, \infty)\,.$$

By the Monotonicity Theorem f must be monotonic (and hence one-to-one) over each of the $m + 1$ intervals

$$(-\infty, c_1]\,, \quad [c_1, c_2]\,, \quad \cdots \quad [c_{m-1}, c_m]\,, \quad [c_m, \infty)\,.$$

The equation $f(x) = 0$ can therefore have at most one solution in each of these $m + 1$ intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most $m + 1$ solutions. The result follows because $m + 1 \leq n + 1$. $\square$

Remark: The alternative solution rests on the Dichotomy Theorem and the Monotonicity Theorem. This machinery is much heavier than that used in the first solution, which rests only on Rolle's Theorem. Indeed, the proof of the Monotonicity Theorem rests on the Mean-Value Theorem, the proof of which rests on Rolle's Theorem.

3. [15] Suppose that $f : (a,b) \to \mathbb{R}$ is twice differentiable and that $f'' : (a,b) \to \mathbb{R}$ is bounded over (a,b). Show that there exists an $M \in \mathbb{R}_+$ such that for all points $x, y \in (a,b)$ one has
$$\big|f'(x) - f'(y)\big| \leq M\,|x-y|\,.$$

Solution: Because $f'' : (a,b) \to \mathbb{R}$ is bounded over (a,b) one has
$$M = \sup\big\{|f''(x)| \,:\, x \in (a,b)\big\} < \infty\,.$$
Let $x, y \in (a,b)$. If $x = y$ then the inequality clearly holds. If $x < y$ then, because f' is differentiable (and hence continuous) over $[x,y]$, the Mean-Value Theorem applied to f' implies there exists a $p \in (x,y)$ such that
$$f'(x) - f'(y) = f''(p)(x-y)\,.$$
Hence,
$$\big|f'(x) - f'(y)\big| = \big|f''(p)\big|\,|x-y| \leq M\,|x-y|\,.$$
The case $y < x$ goes similarly. □

4. [15] Prove that for every $x > 0$ one has
$$1 + \tfrac{3}{2}x < (1+x)^{\frac{3}{2}} < 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2\,.$$

Solution: Define $f(x) = (1+x)^{\frac{3}{2}}$ for every $x > -1$. Then
$$f'(x) = \tfrac{3}{2}(1+x)^{\frac{1}{2}}\,, \qquad f''(x) = \tfrac{3}{4}(1+x)^{-\frac{1}{2}}\,.$$
Let $x > 0$. By the Lagrange Remainder Theorem there exists a $p \in (0,x)$ such that
$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(p)x^2\,.$$
Hence,
$$(1+x)^{\frac{3}{2}} - 1 - \tfrac{3}{2}x = \tfrac{3}{8}(1+p)^{-\frac{1}{2}}x^2\,.$$
Because $p \mapsto (1+p)^{-\frac{1}{2}}$ is a decreasing function while $0 < p < x$, one has the bounds
$$0 < (1+x)^{-\frac{1}{2}} < (1+p)^{-\frac{1}{2}} < 1\,,$$
whereby
$$0 < (1+x)^{\frac{3}{2}} - 1 - \tfrac{3}{2}x < \tfrac{3}{8}x^2\,.$$
The result follows. □

5. [15] Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous over (a, b). Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in (a, b). Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence.

   Solution: Let $\epsilon > 0$. Because $f : (a, b) \to \mathbb{R}$ is uniformly continuous over (a, b), there exists a $\delta > 0$ such that for all points $x, y \in D$ one has
   $$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,.$$
   Because $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for all $k, l \in \mathbb{N}$ one has
   $$k, l > N \quad \Longrightarrow \quad |x_k - x_l| < \delta\,.$$
   Hence, for all $k, l \in \mathbb{N}$ one has
   $$\begin{aligned} k, l > N \quad &\Longrightarrow \quad |x_k - x_l| < \delta \\ &\Longrightarrow \quad \big|f(x_k) - f(x_l)\big| < \epsilon\,. \end{aligned}$$
   Therefore $\{f(x_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. □

   Remark: The characterization of uniform convergence used above was given as assertion (b) of problem 6 below.

6. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D one has
   $$\lim_{k\to\infty} |x_k - y_k| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,."$$

   Solution: "There exist sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D such that
   $$\lim_{k\to\infty} |x_k - y_k| = 0 \quad \text{and} \quad \limsup_{k\to\infty} \big|f(x_k) - f(y_k)\big| > 0\,."$$

   (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ one has
   $$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,."$$

   Solution: "There exists an $\epsilon > 0$ such that for every $\delta > 0$ there exist points $x, y \in D$ such that
   $$|x - y| < \delta \quad \text{and} \quad \big|f(x) - f(y)\big| \geq \epsilon\,."$$