HW 1

1 Proposition 1.1

(a) for every $x \in \mathbf{X}$ there exists $-x \in \mathbf{X}$ such that $x + (-x) = 0$,

$$\begin{aligned}
x + y &= x + z \\
\Rightarrow (-x) + (x + y) &= (-x) + (x + z) \\
\Rightarrow ((-x) + x) + y &= ((-x) + x) + z && \text{by associativity} \\
\Rightarrow 0 + y &= 0 + z && \text{by the property of inverse} \\
\Rightarrow y &= z && \text{by the property of identity}
\end{aligned}$$

(b)

$$\begin{aligned}
x + y &= x \\
\Rightarrow (-x) + x + y &= (-x) + (x) \\
\Rightarrow 0 + y &= 0 \\
\Rightarrow y &= 0
\end{aligned}$$

(c)

$$\begin{aligned}
x + y &= 0 \\
\Rightarrow (-x) + x + y &= (-x) + 0 \\
\Rightarrow 0 + y &= (-x) \\
\Rightarrow y &= (-x)
\end{aligned}$$

(d)

$$\begin{aligned}
(-x) + (-y) + (-(x + y)) + (x + y) &= (-x) + (-y) + (-(x + y)) + (x + y) \\
\Rightarrow (-x) + (-y) + 0 &= ((-x) + x) + ((-y) + y) + (-(x + y)) \\
\Rightarrow (-x) + (-y) &= 0 + 0 + (-(x + y)) \\
\Rightarrow (-x) + (-y) &= -(x + y)
\end{aligned}$$

(e) $-(-x) = -(-x) + 0 = -(-x) + ((-x) + x) = ((-(-x)) + (-x)) + x = 0 + x = x$

2 Proposition 1.2

Since $x \neq 0$ for all the below, inverse x^{-1} of x exists.

(a)

$$\begin{aligned}
xy &= xz \\
\Rightarrow x^{-1}(xy) &= x^{-1}(xz) \\
\Rightarrow (x^{-1}x)y &= (x^{-1}x)z && \text{by associativity} \\
\Rightarrow 1y &= 1z && \text{by the property of inverse} \\
\Rightarrow y1 &= z1 && \text{by commutativity} \\
\Rightarrow y &= z && \text{by the property of identity}
\end{aligned}$$

(b)

$$
\begin{aligned}
xy &= x \\
\Rightarrow x^{-1}(xy) &= x^{-1}x \\
\Rightarrow (x^{-1}x)y &= 1 \\
\Rightarrow 1y &= 1 \\
\Rightarrow y &= 1
\end{aligned}
$$

(c)

$$
\begin{aligned}
xy &= 1 \\
\Rightarrow x^{-1}(xy) &= (x^{-1})1 \\
\Rightarrow (x^{-1}x)y &= x^{-1} \\
\Rightarrow 1y &= x^{-1} \\
\Rightarrow y &= x^{-1}
\end{aligned}
$$

(d)

$$
\begin{aligned}
(xy)^{-1}(xy)x^{-1}y^{-1} &= (xy)^{-1}(xy)x^{-1}y^{-1} \\
\Rightarrow 1x^{-1}y^{-1} &= (xy)^{-1}(xx^{-1})(yy^{-1}) \\
\Rightarrow x^{-1}y^{-1} &= x^{-1}1 \\
\Rightarrow x^{-1}y^{-1} &= xy^{-1}
\end{aligned}
$$

(e) $(x^{-1})^{-1} = (x^{-1})^{-1}1 = (x^{-1})^{-1}((x^{-1})x) = ((x^{-1})^{-1}(x^{-1}))x$

3 Proposition 1.3

(a)

$$
\begin{aligned}
1+0 &= 1 \\
\Rightarrow x(1+0) &= x1 \\
\Rightarrow x1 + x0 &= x1 && \text{by distributivity} \\
\Rightarrow x0 &= 0 && \text{by Prop 1.1(b)}
\end{aligned}
$$

(b) If $x = 0$ then done.
If $x \neq 0$ then there exists a multiplicative inverse x^{-1} of x

$$
\begin{aligned}
xy &= 0 \\
\Rightarrow x^{-1}(xy) &= x^{-1}0 \\
\Rightarrow 1y &= 0 \\
\Rightarrow y &= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
(-x)y + (-(xy)) + xy &= (-x)y + (-(xy)) + xy \\
\Rightarrow ((-x)+x)y + (-(xy)) &= (-x)y + 0 \\
\Rightarrow 0y + (-(xy)) &= (-x)y \\
\Rightarrow 0 + (-(xy)) &= (-x)y \\
\Rightarrow -(xy) &= (-x)y
\end{aligned}
$$

Switch x and y and we get $-(xy) = (-y)x = x(-y)$.

(d) If $(-x) = 0$ then $x + (-x) = 0 \Rightarrow x + 0 = 0 \Rightarrow x = 0$, but $x \neq 0$, hence $(-x) \neq 0$. Inverse of $(-x)$ exists.

$$\begin{aligned}
&(-x)^{-1}(-x) = 1 \\
\Rightarrow\ &(-x)^{-1}(-x)(-x^{-1}) = (-x^{-1}) \\
\Rightarrow\ &(-x)^{-1}(-(-x)(x^{-1})) = (-x^{-1}) \\
\Rightarrow\ &(-x)^{-1}(-(-(xx^{-1}))) = (-x^{-1}) \\
\Rightarrow\ &(-x)^{-1}(xx^{-1}) = (-x^{-1}) \\
\Rightarrow\ &(-x)^{-1}1 = (-x^{-1}) \\
\Rightarrow\ &(-x)^{-1} = (-x^{-1})
\end{aligned}$$

4 Proposition 1.5

(a) $x > 0 \Rightarrow x + (-x) > (-x) \Rightarrow 0 > (-x)$
$x < 0 \Rightarrow x + (-x) < (-x) \Rightarrow 0 < (-x)$

(b) $0 < x \Rightarrow z + 0 < z + x$ by Definition 1.8(i).
Then by transitivity, $y < z < z + x \Rightarrow y < z + x$

$y < z \Rightarrow 0 < z - y$.
Since $x > 0, z - y > 0$, we have $x(z - y) > 0 \Rightarrow xz + x(-y) > 0 \Rightarrow xz > xy$

(c) $x < 0 \Rightarrow y + x < y + 0$ by Definition 1.8(i).
Then by transitivity, $y + x < y < z \Rightarrow y + x < z$

$y < z \Rightarrow 0 < z - y$.
Since $(-x) > 0, z - y > 0$, we have $(-x)(z - y) > 0 \Rightarrow (-x)z + (-x)(-y) > 0 \Rightarrow xy > xz$

(d) For $x > 0$, $xx > 0 \Rightarrow x^2 > 0$
For $x < 0$,$(-x) > 0 \Rightarrow (-x)(-x) > 0 \Rightarrow x^2 = (-x)(-x) > 0$.

(e) First we show $y^{-1} > 0$ for $y > 0$.
By (d), $(y^{-1})^2 > 0$, since $y > 0$, we have $(y^{-1})^2 y > 0 \Rightarrow (y^{-1}) = (y^{-1})(y^{-1})y > 0$

We will prove by contradiction.
Suppose $y^{-1} \geq x^{-1}$, then since $xy > 0$ we get $xy(y^{-1}) \geq xy(x^{-1}) \Rightarrow x \geq y$ which contradicts to $x < y$, so $y^{-1} < x^{-1}$.

5 Proposition 1.6

- proof of transitivity
  If $x, y, z \in \mathbf{X}$, $x < y$ and $y < z$ implies $y - x \in \mathbf{X}^+$ and $z - y \in \mathbf{X}^+$. By positivity properties, we get $(y - x) + (z - y) \in \mathbf{X}^+$. Therefore we have $z - x \in \mathbf{X}^+$ which implies $x < z$.
- proof of trichotomy
  $x < y \Leftrightarrow y - x \in \mathbf{X}^+$
  $y < x \Leftrightarrow -(y - x) = x - y \in \mathbf{X}^+$
  Since exactly one of $y - x \in \mathbf{X}^+$,$-(y - x) \in \mathbf{X}^+$, or $y - x = 0$ is true, exactly one of $x < y$, $y < x$, or $x = y$ is true.

- proof of Definition 1.8 (i)
  If $x < y$ then $y - x \in \mathbf{X}^+$. But then we would have $(y + z) - (x + z) = y + z + (-z) + (-x) = y - x \in \mathbf{X}^+$ which implies $x + z < y + z$.
- proof of Definition 1.8 (ii)
  If $x > 0$ and $y > 0$, we have $x = x - 0 \in \mathbf{X}^+$ and $y = y - 0 \in \mathbf{X}^+$. By positivity properties we get $xy \in \mathbf{X}^+$, which implies $xy > 0$.

  From the first two proofs, we get that $(\mathbf{X}, <)$ is an ordered set. From the last two proofs we get that $\mathbf{X}$ satisfies the definition of an ordered field.

6 Proposition 1.7

(a) For $x > 0$, $|x| = x > 0$.
For $x = 0$, $|x| = 0$.
For $x < 0$, $|x| = -x > 0$ (by Proposition 1.3(a)).
Hence $|x| \geq 0$.

(b) $x = 0 \Rightarrow |x| = 0$ by definition.
If $x \neq 0$, then either $x > 0$ or $x < 0$.
For both cases $|x| > 0$, which means $|x| \neq 0$.
$|x| = 0 \Rightarrow x = 0$

(c) If $x > 0$, then $|x| = x > 0 > -x$.
If $x < 0$, then $|x| = -x > 0 > x$.
$\Rightarrow |x| \geq x$,
$|x| \geq -x$.
Similarly, we have $|y| \geq y$,
$|y| \geq -y$.
Since $|x+y|$ equals to $(x+y)$ or $-(x+y) = (-x)+(-y)$, we have $|x+y| \leq |x|+|y|$ in any case.

(d) If $x = 0$, then $|xy| = |0y| = 0 = |0||y| = |x||y|$.
Same for if $y = 0$.
Divide the rest into four cases

   * $x > 0$, $y > 0$
     Then $xy > 0$
     We get $|x||y| = (x)(y) = (xy) = |xy|$
   * $x > 0$, $y < 0$
     Then $x(-y) > 0 \Rightarrow xy < 0$
     We get $|x||y| = (x)(-y) = -(xy) = |xy|$
   * $x < 0$, $y > 0$
     Then $(-x)y > 0 \Rightarrow xy < 0$
     We get $|x||y| = (-x)(y) = -(xy) = |xy|$
   * $x < 0$,$y < 0$ Then $(-x)(-y) > 0 \Rightarrow xy > 0$
     We get $|x||y| = (-x)(-y) = (xy) = |xy|$

$|x||y| = |xy|$ for all cases, so of course we have $|x||y| \geq |xy|$ for all cases.

(e) If $x > 0$, then $(-x) < 0$, so $|x| = x = -(-x) = |-x|$
If $x = 0$, then $|x| = 0 = |-x|$.
If $x < 0$, then $(-x) > 0$, so $|x| = (-x) = |-x|$.

Therefore we have $|x| = |-x|$ for all x.
By (c) we get $|x| \leq |x-y| + |y|$ and $|y| \leq |y-x| + |x|$
$\Rightarrow |x-y| \geq |x| - |y|$ and $|x-y| = |y-x| \geq |y| - |x|$
If $|x| - |y| = 0$ then $|x-y| \geq 0 = ||x| - |y||$ is automatically true.
If $|x| - |y| > 0$ then use the first equation and we get $|x-y| \geq |x| - |y| = ||x| - |y||$
If $|x| - |y| < 0$ then use the second equation and we get $|x-y| \geq= |y| - |x| = -(|x| - |y|) = ||x| - |y||$

We have then proved $|x-y| \geq ||x| - |y||$.

7 Proposition 1.8
By the first assertion, there exists $k, l \in \mathbf{Z}^+$ such that $-k < x < l$. Then $x \in (-k, l) \subset (-k, l]$. We express $(-k, l]$ as the union of $l+m$ disjoint unit length intervals, $(-k, l] = \cup_{i=-k+1}^{l} (i-1, i]$. Since every interval is disjoint from each other, $x \in (i-1, i]$ for exactly one $i = m$. Since $-k+1 \leq i \leq l$, we have $-k \leq m \leq l$ such that $x \in (m-1, m]$.