1 Pick m=10.
Then $2^{-m} = \frac{1}{1024} < 0.001$,
also $2^{-k} < 2^{-m} < 0.001 \quad \forall k > m$.

2 Negation of $\mathcal{A}(x_k)$ ultimately
$\Rightarrow$ "$\exists m \in \mathbb{N}$ such that $\forall k > m \;\; \mathcal{A}(x_k)$" is not true.
$\Rightarrow \nexists m \in \mathbb{N}$ such that $\forall k > m \;\; \mathcal{A}(x_k)$
$\Rightarrow \forall m \in \mathbb{N}, \exists k > m$ such that "$\mathcal{A}(x_k)$ is not true"
$\Rightarrow \sim \mathcal{A}(x_k)$ frequently.

3 (i) Given any $n \in \mathbb{N} \quad \exists r \in \mathbb{N}$ such that $2\pi r > n + \frac{\pi}{3}$ (By the Archimedean property)
which means $2\pi r - \frac{\pi}{3} > n$
$\cos x > 0.5$ for all $x \in (2\pi r - \frac{\pi}{3}, 2\pi r + \frac{\pi}{3})$
since $(2\pi r + \frac{\pi}{3}) - (2\pi r - \frac{\pi}{3}) = \frac{2}{3}\pi > 1$
By proposition 1.8, $\exists k \in \mathbb{Z}$ such that $k \in (2\pi r - \frac{\pi}{3}, 2\pi r + \frac{\pi}{3})$.
That is, for any $n \in \mathbb{N}$, we can find $k \in \mathbb{N}$ such that $k > 2\pi r - \frac{\pi}{3} > n$, and $\cos k > 0.5$
$\Rightarrow \cos k > 0.5$ frequently as $k \to \infty$

(ii) Given any $n \in \mathbb{N} \quad \exists r \in \mathbb{N}$ such that $2\pi r > n - \frac{\pi}{3}$ (By the Archimedean property)
which means $2\pi r + \frac{\pi}{3} > n$
$\cos x < 0.5$ for all $x \in (2\pi r + \frac{\pi}{3}, 2\pi r + \frac{5\pi}{3})$
since $(2\pi r + \frac{5\pi}{3}) - (2\pi r + \frac{\pi}{3}) = \frac{4}{3}\pi > 1$
By proposition 1.8, $\exists k \in \mathbb{Z}$ such that $k \in (2\pi r + \frac{\pi}{3}, 2\pi r + \frac{5\pi}{3})$.
That is, for any $n \in \mathbb{N}$, we can find $k \in \mathbb{N}$ such that $k > 2\pi r + \frac{\pi}{3} > n$, and $\cos k < 0.5$
$\Rightarrow \cos k < 0.5$ frequently as $k \to \infty$
$\Rightarrow \cos k > 0.5$ not ultimately as $k \to \infty$

4 The first three terms of the subsequence $\{2^{3k}\}$ are $2^0, 2^3, 2^6$.
The first three terms of the subsequence $\{2^{2k+1}\}$ are $2^1, 2^3, 2^5$.

5 ($\Longrightarrow$)
$\forall m \in \mathbb{N}, \exists k > m$ such that $\mathcal{A}(x_k)$
Let $n_1 = 1$.
$\exists n_2 > n_1 = 1$ such that $\mathcal{A}(x_{n_2})$.
Similarly, $\exists n_3 > n_2$ such that $\mathcal{A}(x_{n_3})$.
Inductively, we can get a sequence of indices $\{n_k\}$ such that $n_k < n_{k+1}$ and $\mathcal{A}(x_{n_k})$ for all $k > 0$.
We have found a subsequence $\{x_{n_k}\}$ such that $\mathcal{A}(x_{n_k})$ ultimately as $k \to \infty$.

($\Longleftarrow$)
If $\mathcal{A}(x_{n_k})$ ultimately,
$\Rightarrow \exists m \in \mathbb{N}$ such that $\forall q > m$, we have $\mathcal{A}(x_{n_q})$.
Now, given any $n \in \mathbb{N}$, We can find $k \in \mathbb{N}$ such that $k > \max\{n, m\}$,
then $n_k > n_m \geqslant m$
(Since $n_k - n_{k-1} \geqslant 1 \Rightarrow n_k = \sum_{l=0}^{k-2}(n_{k-l} - n_{k-l-1}) \geqslant (k-1) + n_1 = k$)
Then we have found $n_k \in \mathbb{N}$ such that $n_k > n$ and $\mathcal{A}(x_{n_k})$
This proves that $\mathcal{A}(x_k)$ frequently

6 sequence diverges $\Leftrightarrow$ sequence does not converge
$\Leftrightarrow$ the statement "$\exists a \in \mathbb{R}$ such that $\forall \epsilon > 0$, one has $|a_k - a| < \epsilon$ ultimately as $k \to \infty$"

is false.
$\Leftrightarrow$ $\forall a \in \mathbb{R}$ there exists some $\epsilon_a > 0$ such that the statement "$|a_k - a| < \epsilon_a$ ultimately as $k \to \infty$" is false.
$\Leftrightarrow$ $\forall a \in \mathbb{R}$ there exists some $\epsilon_a > 0$ such that "$|a_k - a| > \epsilon_a$ frequently as $k \to \infty$".

7 (i) $\{a_k\}$ converges to a
Given any $\epsilon > 0$
$\Rightarrow$ $|a_k - a| < \frac{\epsilon}{2}$ ultimately as $k \to \infty$
$\Rightarrow$ $\exists m \in \mathbb{N}$ such that for all $k > m$, we have $|a_k - a| < \frac{\epsilon}{2}$
Similarly,
$\exists m' \in \mathbb{N}$ such that for all $k > m'$, we have $|b_k - b| < \frac{\epsilon}{2}$

Let $M = \max\{m, m'\}$, then
$|(a_k + b_k) - (a + b)| \leqslant |a_k - a| + |b_k - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall k > M$
$\Rightarrow |(a_k + b_k) - (a + b)| < \epsilon$ ultimately as $k \to \infty$
since ϵ is arbitrary,
$\Rightarrow$ $a_k + b_k \to a + b$

(ii) $\{a_k\}$ converges to a
Given any $\epsilon > 0$
$\Rightarrow$ $\exists m \in \mathbb{N}$ such that for all $k > m$, we have $|a_k - a| < \epsilon$
But then,
$|(-a_k) - (-a)| - |a_k - a| < \epsilon \ \ \forall k > m$
$\Rightarrow |(-a_k) - (-a)| < \epsilon$ ultimately as $k \to \infty$
$\Rightarrow$ $-a_k \to -a$

(iv) $\{a_k\}$ converges to a
Given any $\epsilon > 0$
$\Rightarrow$ $\exists m \in \mathbb{N}$ such that for all $k > m$, we have $|a_k - a| < \epsilon$

$\Rightarrow$ $\exists m_1$ such that for all $k > m_1$, we have $|a_k - a| \leqslant \frac{|a|}{2}$,
that is, for $k > m_1$ we have $|a_k| > \frac{|a|}{2}$.

$\exists m_2$ such that for all $k > m_2$, we have $|a_k - a| \leqslant \frac{|a|^2}{2\epsilon}$

Let $m = \max\{m_1, m_2\}$,
then, $|\frac{1}{a_k} - \frac{1}{a}| = |\frac{a_k - a}{a_k a}| \leqslant \frac{2|a_k - a|}{|a|^2} < \epsilon \quad \forall k > m$

(v) Suppose the contrary that $a > b$,
Let $\epsilon = \frac{a-b}{2}$
Since $a_k \to a$ and $b_k \to b$ as $k \to \infty$,
for some m we have $a_k > a - \epsilon$ and $b_k < b + \epsilon$ for all $k > m$.
Then we have $a_k > a - \epsilon = \frac{a+b}{2} = b + \epsilon > b_k$ for all $k > m$,
which contradicts with $a_k \leqslant b_k$ frequently.

So $a \leqslant b$.

(8) $\{a_k\}$ diverges
$\Rightarrow$ $\{a_k\}$ does not converge.
By (i), $\{a_k\}$ is not bounded above (bounded below).
$\Rightarrow$ for all $b \in \mathbb{R}$ and $N \in \mathbb{N}$ $\exists k > N$such that $a_k > b$ ($a_k < b$).

$\Rightarrow$for all $b \in \mathbb{R}$ one has that $a_k > b$ ($a_k < b$) ultimately as $k \to \infty$
$\Rightarrow \lim_{k\to\infty} a_k = \infty \quad (\lim_{k\to\infty} a_k = -\infty)$

(9) (a) If $\liminf\{a_k\} = -\infty$ or $\liminf\{b_n\} = -\infty$, then the inequality is automatically true.
So we are left to consider when both $\liminf\{a_k\} > -\infty$ and $\liminf\{b_k\} > -\infty$ is true.

$\underline{a_k} = \inf\{a_l : l \geqslant k\}$
$\underline{b_k} = \inf\{b_l : l \geqslant k\}$

$\underline{a_k + b_k} = \inf\{a_l + b_l : l \geqslant k\}$

Fix k,
$\underline{a_k} \leqslant a_l \quad \forall l \geqslant k$
$\underline{b_k} \leqslant b_l \quad \forall l \geqslant k$
$\Rightarrow \underline{a_k} + \underline{b_k} \leqslant a_l + b_l \quad \forall l \geqslant k$
$\Rightarrow \underline{a_k} + \underline{b_k}$ is a lower bound of $a_l + b_l$
$\Rightarrow \underline{a_k} + \underline{b_k} \leqslant \underline{a_k + b_k}$ (since $\underline{a_k + b_k}$ is the greatest lower bound of $a_l + b_l$.)
This is true for all $k \in \mathbb{N}$

We can apply Proposition 2.4(i) and get
$\lim \underline{a_k} + \lim \underline{b_k} \leqslant \lim(\underline{a_k + b_k})$
$\Rightarrow \liminf_{k\to\infty} a_k + \liminf_{k\to\infty} b_k \leqslant \liminf_{k\to} a_k + b_k$

By the proof above, given any two sequences $\{a_k\}$ and $\{b_k\}$
we have for sequences $\{-a_k\}$ and $\{-b_k\}$,
$\lim \underline{-a_k} + \lim \underline{-b_k} \leqslant \lim \underline{(-a_k) + (-b_k)} = \lim \underline{-(a_k + b_k)}$

We can observe that $\underline{-a_k} = -\overline{a_k}$

$\Rightarrow \lim(-\overline{a_k}) + \lim(-\overline{b_k}) \leqslant \lim(-\overline{a_k + b_k})$

$\Rightarrow \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k \geqslant \limsup_{\to\infty}(a_k + b_k)$

(b) Since $a_k \leqslant b_k$ ultimately,
$\exists N \in \mathbb{N}$ such that $a_k \leqslant b_k \quad \forall k > N$
Since $\underline{a_k} \leqslant a_k \quad \forall k$
$\Rightarrow \underline{a_k} \leqslant b_k \quad \forall k > N$
$\Rightarrow \underline{a_k} \leqslant \underline{b_k} \quad \forall k > N$ (Since $\underline{b_k}$ is the greatest lower bound for $b_l,\ l > k$)
Taking the limit on both sides,we get
$\lim \underline{a_k} \leqslant \lim \underline{b_k} \Rightarrow \liminf_{k\to\infty} a_k \leqslant \liminf_{k\to\infty} b_k$

Similarly, we can prove $\limsup_{k\to\infty} a_k \leqslant \limsup_{k\to\infty} b_k$.

(10) From $n_1 < n_2 < \cdots < n_k$ we can observe that $k \leqslant n_k$
$\therefore \{a_l|\ l \geqslant k\} \supset \{a_l|\ l \geqslant n_k\}$
$\underline{a_k} \leqslant \underline{a_{n_k}}$
taking the limit on both sides,we have

$\lim \underline{a_k} \leqslant \lim \underline{a_{n_k}} \Rightarrow \liminf_{k\to\infty} a_k \leqslant \liminf_{k\to\infty} a_{n_k}$

To prove $\limsup_{k\to\infty} a_{n_k} \leqslant \limsup_{k\to\infty} a_k$,
we substitute the sequences $\{-a_k\}$ and $\{-a_{n_k}\}$ into the previous inequality obtained.

(11) For a bounded sequence $\{a_k\}$, we have
$\limsup_{k\to\infty} a_k < \infty$.
By the second part of Proposition 2.8,
there exists a subsequence $\{a_{n_k}\}$ such that
$\lim_{k\to\infty} a_{n_k} = \limsup_{k\to\infty} a_k < \infty$
Then $\{a_{n_k}\}$ is a convergent subsequence.
We have proved that a convergent subsequence always exists.

(12) Let $s_n = \sum_{k=0}^{n} a_k$.
$\sum_{k=o}^{\infty}$ converges implies that $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{R}$.
$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$

(13) (a) There does not exist an example except for the case where $b_k = 0$ ultimately as $k \to \infty$
If we don't have $b_k = 0$ ultimately then $a_k \leqslant Mb_k$ ultimately implies $\frac{a_k}{b_k} \leqslant M$ ultimately.
$\Rightarrow \limsup_{k\to\infty} \frac{a_k}{b_k} = M < \infty$
So if direct comparison test applies, limit comparison test also applies.

(b) Let$a_k = \frac{1}{k^2}$ and let $b_k = \frac{1}{(k-1)^2}$
$\limsup_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}(\frac{k-1}{k})^2 = 1 < \infty$
but $\frac{a_{k+1}}{a_k} = (\frac{k}{k+1})^2 > (\frac{k-1}{k})^2 = \frac{b_{k+1}}{b_k} \quad \forall k$
So here, the limit comparison test applies, but the ratio comparison test fails.

(14) Let $a_k = 1$, then $\limsup \sqrt[k]{a_k} = 1$
$\sum_{k=0}^{\infty} a_k$ diverges.

Let $b_k = \frac{1}{k^2}$, then $\limsup \sqrt[k]{b_k} = 1$
$\sum_{k=0}^{\infty} b_k$ converges.

(15) proven in class

(16) Let $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = r$
Pick any $\rho < r$,
$\exists N \in \mathbb{N}$ such that $\rho < \frac{a_{k+1}}{a_k} \quad \forall k > N$
$\Rightarrow a_{k+1} \geqslant \rho a_k \quad \forall k > N$
$\Rightarrow a_k \geqslant \rho^{k-N} a_N \quad \forall k > N$
$\Rightarrow \sqrt[k]{a_k} \geqslant \sqrt[k]{\rho^{k-N} a_N} = \sqrt[k]{\rho^k}\sqrt[k]{\frac{a_N}{\rho^N}} = \rho\sqrt[k]{c}$, where c is a constant.
$\liminf_{k\to\infty} \sqrt[k]{a_k} \geqslant \rho\sqrt[k]{c} = \rho$
Since ρ can be taken arbitrary close to r
$\liminf_{k\to\infty} \sqrt[k]{a_k} \geqslant r = \liminf_{k\to\infty} \frac{a_{k+1}}{a_k}$
we have proven the left inequality.
The right inequality is proven the same way.