HW 3

(1) Proposition 2.12
By the definition of a Cauchy sequence, we know that
for any $\epsilon > 0$, there exists N_ϵ such that
$k, l \geqslant N_\epsilon \Rightarrow |a_k - a_l| < \epsilon$.

Let $\epsilon = 1$. Then there is an N such that
$k \geqslant N \Rightarrow |a_N - a_k| < 1$,
which means $|a_k| < |a_N| + 1$ for all $k > N$.
Let $M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a_N| + 1\}$
then it is clear that $|a_k| \leqslant M$ for all $k \in \mathbb{N}$
$\Rightarrow \{a_k\}$ is bounded.

(2) Proposition 3.6 (Cauchy 2^k test)
a_k is nonincreasing, so for $k \leqslant 2^j \Rightarrow a_k \geqslant a_{2^j}$
for $k \geqslant 2^{j-1} \Rightarrow a_k \leqslant a_{2^{j-1}}$
therefore we have
$a_{2^j} \leqslant a_k \leqslant a_{2^{j-1}}$ for $2^{j-1} \leqslant k < 2^j$

let $b_k = 2a_{2^j}$ for $2^{j-1} \leqslant k < 2^j$, $j \in \mathbb{N}$
$\sum_{k=1}^{\infty} b_k = \sum_{j=1}^{\infty} \sum_{k=2^{j-1}}^{2^j} b_k = 2 \cdot \sum_{j=1}^{\infty} 2^{j-1} a_{2^j} = \sum_{j=1}^{\infty} 2^j a_{2^j}$
Since $b_k = 2a_{2^j} \leqslant 2a_k$
By direct comparison,
$\sum_{k=1}^{\infty} b_k$converges if $\sum_{k=1}^{\infty} a_k$converges
So $\sum_{j=1}^{\infty} 2^j a_{2^j}$ converges if $\sum_{k=1}^{\infty} a_k$ converges.

For the second half, we do similar arguments.
Let $c_k = a_{2^{j-1}}$ for $2^{j-1} \leqslant k < 2^j$, $j \in \mathbb{N}$
then $\sum_{k=1}^{\infty} c_k = \sum_{j=1}^{\infty} \sum_{k=2^{j-1}}^{2^j} c_k = \sum_{j=1}^{\infty} \sum_{k=2^{j-1}}^{2^j} a_{2^{j-1}} = \sum_{j=1}^{\infty} 2^{j-1} a_{2^{j-1}} = \sum_{j=0}^{\infty} 2^j a_{2^j}$
Since $a_k \leqslant a_{2^{j-1}} = c_k$ for $2^{j-1} \leqslant k \leqslant 2^j$ and all $j \in \mathbb{N}$,
$\sum_{k=1}^{\infty} a_k$ converges if $\sum_{k=1}^{\infty} c_k$ converges.
$\sum_{k=1}^{\infty} 2^j a_{2^j}$ converges if $\sum_{k=1}^{\infty} c_k$ converges.

(3) $\overline{s} = \lim_{k\to\infty} S_{2k}$
$\underline{s} = \lim_{k\to\infty} S_{2k+1}$
we want to show that if $\overline{s} = \underline{s}$, then $\lim_{k\to\infty} S_k = \overline{s} = \underline{s}$

With $\overline{s} = \lim_{k\to\infty} S_{2k}$ and $\underline{s} = \lim_{k\to\infty} S_{2k+1}$, we get
given any $\epsilon > 0$
$\exists N_1$ such that $|S_{2k} - \overline{s}| < \epsilon \quad \forall k > N_1$
$\exists N_2$ such that $|S_{2k+1} - \underline{s}| < \epsilon \quad \forall k > N_2$,
then $|S_k - \underline{s}| = |S_k - \overline{s}| < \epsilon \quad \forall k > \max\{2N_2 + 1, 2N_1\}$
Since $\epsilon > 0$ is arbitrary,
$\lim_{k\to\infty} S_k = \overline{s} = \underline{s}$.

(4) There are many ways to determine convergence for every problem. The answers here are just for reference.

(a) We can use the ratio test for positive series here
$\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \limsup_{k\to\infty} \frac{a^{k+1}}{(k+1)^p} \cdot \frac{k^p}{a^k} = \limsup_{k\to\infty} \left(\frac{k}{k+1}\right)^p a$
for $a > 1$, $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \quad \Rightarrow \sum_{k=1}^{\infty} \frac{a^k}{k^p}$ diverges.
for $a < 1$, $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \quad \Rightarrow \sum_{k=1}^{\infty} \frac{a^k}{k^p}$ converges.
for $a = 1$, the series becomes $\sum_{k=1}^{\infty} \frac{1}{k^p}$, which converges when $p > 1$, and diverges when $p \leqslant 1$.

(b) We can apply direct comparison test with the series $\sum_{k=1}^{\infty} \frac{1}{5k}$
Since $\frac{1}{5k} \leqslant \frac{1}{2k+3}$ for all $k \geqslant 1$,
and since $\sum_{k=1}^{\infty} \frac{1}{5k}$ diverges (it is a multiple of the harmonic series),
$\sum_{k=1}^{\infty} \frac{1}{2k+3}$ diverges.

(c) Since $\lim_{k\to\infty} \frac{1}{k} = 0$
By Proposition 3.9 for alternating series, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

(d) Since $\frac{1}{k(k+1)} \leqslant \frac{1}{k^2}$ for all $k \geqslant 1$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges,
by direct comparison test we have that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges.

(e) We can use the ratio test here.
$\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \limsup_{k\to\infty} \frac{k+1}{k} \cdot \frac{e^{-(k^2+2k+1)}}{e^{-k^2}} = \limsup_{k\to\infty} \frac{k+1}{k} e^{-(2k+1)} = 0 < 1$
Thus, the series converges.

(f) Since $\left(\frac{k+1}{k^2+1}\right)^3 \leqslant \left(\frac{2k}{k^2}\right)^3 = \frac{8}{k^3}$ for all $k \geqslant 1$,
and $\sum_{k=1}^{\infty} \frac{8}{k^3}$ converges.
Using the direct comparison test,
$\sum_{k=1}^{\infty} \left(\frac{k+1}{k^2+1}\right)^3$ converges.

(g) $\lim_{k\to\infty} k \sin\left(\frac{1}{k}\right) = 1$
From Proposition 3.1, $\sum_{k=0}^{\infty} k \sin\left(\frac{1}{k}\right)$ cannot converge.

(5) $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \limsup_{k\to\infty} \left(\frac{k+1}{k}\right)^\alpha \frac{e^k}{e^{k+1}} = \frac{1}{e} < 1$
From the ratio test, we know that the series converges.

(6) By the integral test
$\sum_{k=1}^{\infty} \frac{1}{(k+1)[\ln(k+1)]^\alpha}$ converges if and only if $\int_1^\infty \frac{1}{(x+1)[\ln(x+1)]^\alpha} dx$ converges.
for $\alpha = 1$, $\int_1^\infty \frac{1}{(x+1)[\ln(x+1)]^\alpha} dx = \ln(\ln(x+1))\Big|_{x=1}^{\infty} = \infty$
for $\alpha \neq 1$, $\int_1^\infty \frac{1}{(x+1)[\ln(x+1)]^\alpha} dx = \frac{[\ln(x+1)]^{1-\alpha}}{1-\alpha}\Big|_{x=1}^{\infty}$
when $\alpha < 1$ the integral diverges, and when $\alpha > 1$ the integral converges.
The series converges if and only if $\alpha > 1$.