HW 6

1. 3.1 problem 1

   (a) False
   Let
   $$f(x) = \begin{cases} 0 & \text{for } x > 0 \\ 1 & \text{for } x \leqslant 0 \end{cases}$$
   $$g(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leqslant 0 \end{cases}$$
   Here, $f(x)$ and $g(x)$ are not continuous functions, but $(f+g)(x) = 1$ is a continuous function.

   (b) False
   Let
   $$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leqslant 0 \end{cases}$$
   Here, $f^2(x) = 1$ is a continuous function, but $f(x)$ is not a continuous function.

   (c) True
   Since g is continuous $-g$ is continuous. Since the sum of continuous functions is continuous, we have $f = (f+g) + (-g)$ is continuous.

   (d) True
   Given any convergent sequence $\{x_k\}$ in $\mathbb{N}$ converging to $x \in \mathbb{N}$, we have $x_k = x$ ultimately. So, $f(x_k) = f(x)$ ultimately. Hence, every function $f : \mathbb{N} \to \mathbb{R}$ is continuous.

2. 3.1 problem 3
   $f(x)$ is continuous in $\mathbb{R}\backslash\{0\}$.

   For any fixed $x > 0$ and given ϵ, we know $x^2 - y^2 = (x+y)(x-y)$.
   By letting $\delta = \min\{\frac{\epsilon}{2|x|+2}, 1\}$
   $|x - y| < \delta \Rightarrow |y| < |x| + 1, \quad |x^2 - y^2| < (2|x|+1)(\delta) < \frac{2|x|+1}{2|x|+2} \cdot \epsilon < \epsilon$
   f is continuous at x where $x < 0$.

   For any fixed $x < 0$ and given $\epsilon > 0$, we know $|(x+1) - (y-1)| = |x - y|$,
   so $|(x+1) - (y+1)| < \epsilon$ if $|x - y| < \epsilon$
   f is continuous at x where $x > 0$.

   Consider a sequence $\{x_k\} \subset \mathbb{R}^+$, $x_k \to 0$
   $\lim_{k\to\infty} f(x_k) = 1 \neq 0 = f(0)$
   $\Rightarrow f(x)$ is not continuous at 0.

3. 3.1 problem 5
   To show that $f(x)$ is continuous at $x > 0$ and $x < 0$ is similar to the previous problem.

   To show that $f(x)$ is continuous at $x = 0$,
   Consider any sequence $x_k \to 0$,
   $|x_k| < 1$ ultimately as $k \to \infty$

$|x_k^2| < |x_k|$ ultimately.
$|f(x_k)| < |x_k|$ ultimately as $k \to \infty$.
$f(x_k) \to 0 = f(0)$ as $k \to \infty$
$f(x)$ is continuous at $x = 0$.

4. 3.1 problem 13
   Let $\{u_k\}$ be a sequence in D, and $u_k \to u \in D$
   Then $|f(u_k) - f(u)| < c|u_k - u|$,
   by the comparison lemma we have $f(u_k) \to f(u)$ as $k \to \infty$
   Since $f(u_k) \to f(u)$ for all $u_k \to u$ in D
   f is continuous on D.

5. 3.3 problem 1

   a. False.
      Let $f(x) = \sin x$, $f(\mathbb{R}) = [-1, 1] \neq \mathbb{R}$

   b. False.
      Since f is not continuous, we can assign f as
      $$f(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ 0 & x \in (\frac{1}{2}, 1] \end{cases}$$
      $f([0, 1]) = 0, 1$ is not an interval

   c. False.
      Let $D = (0, 1) \cup (2, 3)$, $f(x) = x$,
      then $f(D) = (0, 1) \cup (2, 3)$ is not an interval but is continuous

   d. True.
      f is continuous and $[0, 1]$ is an interval.
      By Theorem 3.14, $f([0, 1])$ is an interval.
      $f(0)$ and $f(1)$ must be in the interval
      And since f is strictly increasing, $f(0) < f(x) < f(1)$ for all $0 < x < 1$.
      So $f(0)$ and $f(1)$ are the endpoints of $f([0, 1])$,
      $\Rightarrow f([0, 1]) = [f(0), f(1)]$

6. 3.3 problem 6
   Let $g(x) = f(x) - x$,
   Since $f(x)$ is bounded, $\exists M > 0$ such that $-M < f(x) < M$
   then $g(x) < 0$ for any $x > M$, $g(y) > 0$ for any $y < -M$.
   By the intermediate value theorem,
   $\exists z, \quad y < z < x$ such that $g(z) = 0$
   $\Rightarrow \exists z \in \mathbb{R}$ such that $f(z) = z$.

7. 3.3 problem 10
   Suppose that f is not a constant function.
   $\exists r_1 < r_2$ such that $f(r_1) \neq f(r_2)$.
   WLOG, suppose $f(r_1) < f(r_2)$,
   By the density of irrational numbers, there exists an irrational number c such that $f(r_1) < c < f(r_2)$.
   By intermediate value theorem,
   $\exists x,\ r_1 < x < r_2$ such that $f(x) = c$

Then this contradicts to our assumption that the image of f consists entirely of rational numbers.

8. 3.4 problem 5
   Let $u_n = n$ and $v_n = n + \frac{1}{n}$
   Then $|u_n - v_n| = \frac{1}{n} \to 0$ as $n \to \infty$,
   But $|f(u_n) - f(v_n)| = |u_n^3 - v_n^3| = [3n + 3\frac{1}{n} + \frac{1}{n^3}] \to \infty$ as $n \to \infty$
   f is not uniformly continuous.

9. 3.4 problem 8
   Let $f(x) = \frac{1}{x-a}$,
   Let $u_n = a + \frac{1}{n}$ and $v_n = a + \frac{1}{n+1}$,
   then $|u_n - v_n| = \frac{1}{n} - \frac{1}{n+1} \to 0$ as $n \to \infty$
   But $|f(u_n) - f(v_n)| = n + 1 - n = 1 \nrightarrow 0$ as $n \to \infty$
   $f(x)$ is not uniformly continuous on $I = (a, b)$

10. 3.4 problem 10
    For any two sequences u_n and v_n such that $|u_n - v_n| \to 0$ as $n \to \infty$
    We have $|f(u_n) - f(v_n)| \leqslant C|u_n - v_n| \to 0$ as $n \to \infty$.
    So f is uniformly continuous.

11. Prove theorem 3.22
    (i $\Rightarrow$ ii)
    Suppose the $\epsilon - \delta$ criterion is not true.
    Then $\exists \epsilon > 0$ such that $\forall \delta > 0$,
    there is $|u - v| < \delta$ but $|f(u) - f(v)| \geqslant \epsilon$
    let $\delta_n = \frac{1}{n}$
    Then we can find two sequences u_n, v_n such that
    $|u_n - v_n| < \delta_n$, but $|f(u_n) - f(v_n)| \geqslant \epsilon$ for all n,
    So we have $\lim_{n\to\infty}[u_n - v_n] = 0$, but $\lim_{n\to\infty}[f(u_n) - f(v_n)] \neq 0$ which contradicts (i.)
    So the $\epsilon - \delta$ criterion must be satisfied if (i) is true.

    (ii $\Rightarrow$ i)
    Let u_n, v_n be sequences such that $u_n - v_n \to 0$
    By (ii), given ϵ, $\exists \delta$ such that $|f(u_n) - f(v_n)| < \epsilon$ if $|u_n - v_n| < \delta$.

    Also $\exists N \in \mathbb{N}$ such that $|u_n - v_n| < \delta$ for all $n > N$
    then for this N, $|f(u_n) - f(v_n)| < \epsilon$ for all $n > N$
    So $\lim_{n\to\infty}[f(u_n) - f(v_n] = 0$ if $\lim_{n\to\infty}[u_n - v_n] = 0$

12. 3.6 problem 7
    Let $x < y$
    Case 1. $0 < x < y$
    $(y^n - x^n) = (y - x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1})$
    $(y - x) > 0$, and $y^{n-1-k}x^k > 0$ for all $k = 1, \cdots, n-1$ since $x > 0, y > 0$

    Case 2. $x < 0 < y$
    $x^n < 0 < y^n$

Case 3. $x < y < 0$
then $-x > -y > 0$
$y^n - x^n = ((-x)^n - (-y)^n)$ since n is odd
$((-x)^n - (-y)^n) > 0$ from case 1.

$y^n > x^n$ for all $y > x$
$f(x) = x^n$ is strictly increasing

Since f is continuous, and strictly increasing, by similar arguments with the ones in 3.3 problem 1.d., we have $f([a,b]) = [f(a), f(b)]$
Since for any $N \in \mathbb{N}$,
$\exists b > N,\ a < -N$ such that $b^n > N^n > N,\ a^n < (-N)^n < -N$,
for any $N \in \mathbb{N}$, $\exists a, b$ such that $f([a,b]) = [f(a), f(b)] = [a^n, b^n] \supset [-N, N]$
$\Rightarrow f(\mathbb{R}) = \mathbb{R}$

13. 3.6 problem 11
We know from (3.26) that $(x^n)^m = x^{nm} = x^{mn} = (x^m)^n$ for all integers m, n
We also have $(y^n)^{1/n} = y = (y^{1/n})^n$ for all $y > 0$ and integer n
Then $(x^{1/n})^m = (((x^{1/n})^m)^n)^{1/n} = ((x^{1/n})^{mn})^{1/n} = (((x^{1/n})^n)^m)^{1/n} = (x^m)^{1/n}$