HW 7

1. 3.7 Problem 2

   a. Let $\{x_n\}$ be a sequence converging to 1, with $x_n \neq 1$ for all $n$.
   $\lim_{n\to\infty} \frac{x_n^4-1}{x_n-1} = \lim_{n\to\infty}[x_n^3 + x_n^2 + x_n + 1] = 4$
   Since this is true for all sequences $\{x_n\} \subset \mathbb{R}\backslash\{1\}$ converging to 1,
   We get $\lim_{x\to 1} \frac{x^4-1}{x-1} = 4$.

   b. Let $\{x_n\}$ be a sequence converging to 1, with $x_n \neq 1$ for all $n$.
   $\lim_{n\to\infty} \frac{\sqrt{x_n}-1}{x_n-1} = \lim_{n\to\infty} \frac{1}{\sqrt{x_n}+1} = \frac{1}{2}$
   Since this is true for all sequences $\{x_n\} \subset \mathbb{R}\backslash\{1\}$ converging to 1,
   We get $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2}$.

2. 3.7 Problem 4

   a. Let $\{x_n\}$ be a sequence converging to 0, with $x_n \neq 0$ for all $n$.
   $\lim_{n\to\infty} \frac{1+1/x_n}{1+1/x_n^2} = \lim_{n\to\infty} \frac{(x_n+1)x_n}{x_n^2+1} = \frac{(1)(0)}{1} = 0$
   Since this is true for all sequences $\{x_n\} \subset \mathbb{R}\backslash\{0\}$ converging to 0,
   We get $\lim_{x\to 0} \frac{1+1/x}{1+1/x^2} = 0$.

   b. Let $\{x_n\}$ be a sequence converging to 0, with $x_n \neq 0$ for all $n$.
   $\lim_{n\to\infty} \frac{1+1/x_n^2}{1+1/x_n} = \lim_{n\to\infty} \frac{(x_n^2+1)}{(x_n+1)x_n} = \infty$
   Since this is true for all sequences $\{x_n\} \subset \mathbb{R}\backslash\{0\}$ converging to 0,
   We get $\lim_{x\to 0} \frac{1+1/x^2}{1+1/x} = \infty$.

   c. Let $\{x_n\}$ be a sequence converging to 0, with $x_n \neq 0$ for all $n$.
   $\lim_{n\to\infty} \frac{1+1/(x_n-1)}{2+1/(x_n-1)^2} = \lim_{n\to\infty} \frac{(x_n-1+1)(x_n-1)}{2(x_n-1)^2+1} = \frac{(1)(0)}{1} = 0$
   Since this is true for all sequences $\{x_n\} \subset \mathbb{R}\backslash\{0\}$ converging to 0,
   We get $\lim_{x\to 0} \frac{1+1/(x-1)}{2+1/(x-1)^2} = 0$.

3. 3.7 Problem 6
   Not necessarily.
   Take $D = \{0\}$,(the set that consists of only the point 0)
   Then $\sup D = 0$, but 0 is not a limit point of $D$.
   ($D$ has only an isolated point)

4. 3.7 Problem 8

   a. Given a point $x_0$ in $D$.
   Either $\exists r$ such that $\forall x \in D\backslash\{x_0\}$, $x \notin (x_0 - r, x_0 + r)$,
   or $\forall r > 0$, $\exists\ x$ such that $x \in D\backslash\{0\}$ and $x \in (x_0 - r, x_0 + r)$
   In the first case, $x_0$ is an isolated point.
   In the second case, we can find $\{x_k\} \subset D\backslash\{x_0\}$ such that $x_k \in (x_0 - \frac{1}{2^k}, x_0 + \frac{1}{2^k})$
   Then, $x_k \to x_0$, so $x_0$ is a limit point.
   So $x_0$ is either an isolated point or a limit point.

   b. $x_0$ is an isolated point $\Rightarrow \exists\ r > 0$ such that for all $x \in D\backslash\{x_0\}$ , $x \notin (x_0-r, x_0+r)$.
   $\Rightarrow$ for any $x_k \to x_0$, $x_k = x_0$ ultimately as $k \to \infty$
   and so $f(x_k) = f(x_0)$ ultimately as $k \to \infty$
   $\Rightarrow f(x_k) \to f(x_0)$ for $x_k \to x_0$
   $\Rightarrow f$ is continuous at $x_0$.

c. Since $x_0$ is a limit point, there exists $\{x_k\} \subset D\backslash\{x_0\}$ such that $x_k \to x_0$.

$\lim_{x\to x_0} f(x) = f(x_0)$
$\Leftrightarrow \lim_{k\to\infty} f(x_k) \to f(x_0)$ for all $x_k \to x_0$, where $\{x_k\} \subset D\backslash\{x_0\}$
$\Leftrightarrow \lim_{k\to\infty} f(x_k) \to f(x_0)$ for all $x_k \to x_0$ (Since $f(x_k) = f(x_0)$ for $x_k = x_0$)
$\Leftrightarrow f$ is continuous at $x_0$.

5. 3.7 Problem 12
Let $M = \inf_{x\in(a,b)}\{f(x)\}$
for any $\epsilon > 0$ there exists $x \in (a,b)$ such that $f(x) < M + \epsilon$
Given any sequence $\{x_k\}$ converging to $a$,
Since $x > a$
$\exists N \in \mathbb{N}$ such that $x_k < x$ for all $k > N$
then since $f$ ia monotone, $f(x_k) \leqslant f(x) < M + \epsilon$ for all $k > N$
also $M \leq f(x_k)$ for all $k$
$\Rightarrow \forall\epsilon > 0 \ \ \exists N \in \mathbb{N}$ such that $M \leq f(x_k) < M + \epsilon$ for all $k > N$
By the definition of limit, $\lim_{k\to\infty} f(x_k) = M$
$\Rightarrow \lim_{x\to a} f(x)$ exists and equals $\inf_{x\in(a,b)}\{f(x)\}$

6. 4.1 Problem 4

a. $f'(x) = \lim_{x\to 1} \frac{\sqrt{x+1}-\sqrt{2}}{x-1} = \lim_{x\to 1} \frac{\sqrt{x+1}-\sqrt{2}}{(x+1)-(2)} = \lim_{x\to 1} \frac{1}{\sqrt{x+1}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$

b. $f'(x) = \lim_{x\to 1} \frac{x^3+2x-3}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+3)}{(x-1} = \lim_{x\to 1}[x^2+x+3] = 5$

c. $f'(x) = \lim_{x\to 1} \frac{1/(1+x^2)-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{1-x^2}{2(1+x^2)(x-1)} = \lim_{x\to 1} \frac{-(1+x)}{2(1+x^2)} = -\frac{1}{2}$

7. 4.1 Problem 6
Let $\{x_k\} \subset J\backslash\{x_0\}$ be a sequence converging to $x_0$
By continuity of $h$ and the property $h(x) \neq h(x_0)$ for $x \neq x_0$,
we have the sequence $\{h(x_k)\} \subset I\backslash\{h(x_0)\}$ converging to $h(x_0)$
Since $f$ is differentiable at $h(x_0)$,
$f'(h(x_0)) = \lim_{h(x_k)\to h(x_0)} \frac{f(h(x_k))-f(h(x_0))}{h(x_k)-h(x_0)}$
which is equal to $\lim_{x_k\to x_0} \frac{f(h(x_k))-f(h(x_0))}{h(x_k)-h(x_0)}$
Since this is true for all sequences $\{x_k\} \subset J\backslash\{x_0\}$ converging to $x_0$,
we have $f'(h(x_0)) = \lim_{x\to x_0} \frac{f(h(x))-f(h(x_0))}{h(x)-h(x_0)}$

8. 4.1 Problem 7

a. Let $h(x) = 1 + x$, $x_0 = 0$.
We see that $(x+1) \neq (x_0+1)$ for all $x \neq x_0$,
Exercise 6 can be applied to get the result.

b. Let $h(x) = \sqrt{x}$, $x_0 = 1$.
We see that $\sqrt{x} \neq \sqrt{x_0}$ for all $x \neq x_0$, $x > 0$
Exercise 6 can be applied to get the result.

c. Let $h(x) = x^2$, $x_0 = 1$.
We see that $x^2 \neq x_0^2$ for all $x \neq x_0$, $x > 0$
Exercise 6 can be applied to get the result.

d. $\lim_{x\to 1} \frac{f(x^2)-f(1)}{x-1} = \lim_{x\to 1} \frac{f(x^2)-f(1)}{x^2-1}(x+1) = 2f'(1)$
where the last equality is obtained by using Exercise 6.

e. $\lim_{x\to 1} \frac{f(x^3)-f(1)}{x-1} = \lim_{x\to 1} \frac{f(x^3)-f(1)}{x^3-1}(x^2+x+1) = 3f'(1)$
where the last equality is obtained by using Exercise 6.

9. 4.1 Problem 9
$0 \leqslant f(0) \leqslant 0 \Rightarrow f(0) = 0$
$\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$
$-|x|^2 \leqslant |f(x)| \leqslant |x|^2$
$\Rightarrow -|x| \leqslant |\frac{f(x)}{x}| \leqslant |x|$
Limit of $\frac{f(x)}{x}$ exists.
$\Rightarrow f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = 0$

10. 4.1 Problem 18
$f(0) = 1 + 4 \cdot 0 + 0 \cdot h(0) = 1$

$f'(0) = \lim_{x\to 0} \frac{(1+4x+x^2h(x))-1}{x-0} = \lim_{x\to 0}[4 + xh(x)]$
Since $h$ is bounded, $\exists M > 0$ such that $-M \leqslant |h(x)| \leqslant M$
$\Rightarrow -|x|M \leqslant |xh(x)| \leqslant |x|M$
$\Rightarrow \lim_{x\to 0} xh(x) = 0$
$\Rightarrow f'(0) = \lim_{x\to 0}[4 + xh(x)] = 4$

11. Let $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$. Then we have $\lim_{x\to x_0}(f+g)(x) = L+M$, $\lim_{x\to x_0}(fg)(x) = LM$, $\lim_{x\to x_0} \frac{f}{g}(x) = \frac{L}{M}$

    See Theorem 3.36 of text.

12. See Theorem 4.6