HW 8.

1. 4.3 Problem 1

   a. False.
   Let $f(x) = x^3$, $f(x)$ is strictly increasing, but $f'(0) = 0$

   b. True.
   Since f is nondecreasing, we have $f(x) \leqslant f(y)$ if $x < y$.
   So $\frac{f(x)-f(x_0)}{x-x_0} \geqslant 0$ for all $x, x_0 \in \mathbb{R}$
   By Lemma 2.21, $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geqslant 0$
   $f'(x_0) \geqslant 0$ for all points x_0 in $\mathbb{R}$

   c. True.
   Since $f(0) \geqslant f(x)$ for all $x \in [-1, 1]$,
   $\frac{f(x)-f(0)}{x-0} \leqslant 0$ for $x > 0$
   So $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \leqslant 0$
   $\frac{f(x)-f(0)}{x-0} \geqslant 0$ for $x < 0$
   So $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} \geqslant 0$
   Since derivative exists, both limits are equal.
   $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$

   d. False
   Let $f(x) = x$
   $f(1) \geqslant f(x)$ for all $x \in [-1, 1]$
   but $f'(1) = 1 \neq 0$

2. 4.3 Problem 4
   $f'(x) = 3x^2 - 3 = 3(x^2 - 1) < 0$ for $0 < x < 1$
   $\Rightarrow f(x) > f(y)$ for all $0 < x < y < 1$
   If $f(x)$ has two solutions in $(0, 1)$, then we would have $f(x) = f(y) = 0$ for some $0 < x < y < 1$, which contradicts the previous statement.

3. 4.3 Problem 7
   You can use Rolle's Theorem as follows.
   $f'(x) = nx^{n-1} + a$
   $f''(x) = n(n-1)x^{n-2} > 0$ for all $x \in \mathbb{R}$ since n is even $\Rightarrow f'$ is strictly increasing.
   If f has three or more zeros,
   $\exists a, b, c$ such that $f(a) = f(b) = f(c) = 0$.
   Then by Rolle's Theorem $\exists x \in (a, b)$ and $y \in (b, c)$
   such that $f'(x) = 0$ and $f'(y) = 0$
   But f' cannot have two distinct zeros since f' is strictly increasing.
   Therefore, f has at most two zeros.

   If n is odd, there could be one or three solutions depending on the values of a, b.
   If $a > 0$, then $f'(x) > 0$ for all x, $f(x)$ is strictly increasing. So f(x) has exactly one solution in this case.
   If $a = 0$, then $f(x) = x^n + b$ is strictly increasing. $f(x)$ has only one solution.
   If $a < 0$ then $f(x)$ is increasing in the intervals $(-\infty, -(\frac{-a}{n})^{\frac{1}{n-1}})$ and $((\frac{-a}{n})^{\frac{1}{n-1}}, \infty)$, decreasing in the interval $(-(\frac{-a}{n})^{\frac{1}{n-1}}, (\frac{-a}{n})^{\frac{1}{n-1}})$.
   So depending on b, $f(x)$ can have one or three solutions.

4. 4.3 Problem 11
   Suppose the contrary. (Suppose f has $n+1$ or more solutions)
   Then $\exists a_1 < a_2 < \cdots < a_{n+1}$ such that
   $f(a_1) = f(a_2) = \cdots = f(a_{n+1}) = 0$
   Then by Rolle's theorem, there exists $x_1, x_2, \cdots, x_n$
   such that $a_1 < x_1 < a_2 < x_2 < a_3 < \cdots < a_n < x_n < x_{n+1}$ and
   $f'(x_1) = f'(x_2) = \cdots = f'(x_n) = 0$,
   but this contradicts to the fact that f' has at most $n-1$ zeros.
   So f can have at most n solutions (zeros).

5. 4.4 Problem 3

   a. $f'(t) = 2t$, $g'(t) = 3t^2$
      $\frac{f(1)-f(0)}{g(1)-g(0)} = \frac{1-0}{1-0}$
      $\frac{f'(c)}{g'(c)} = \frac{2c}{3c^2} = \frac{2}{3c}$
      so if $c = \frac{2}{3}$,we have
      $\frac{f(1)-f(0)}{g(1)-g(0)} = \frac{f'(c)}{g'(c)}$
   b. If $f(1) - f(0) = f'(c)(1-0)$, then $f'(c) = 1$, $\Rightarrow c = \frac{1}{2}$
      If $g(1) - g(0) = g'(0)(1-0)$, then $g'(c) = 1$, $\Rightarrow c = \frac{1}{\sqrt{3}}$
      There is no c that satisfies both equations.

6. 4.4 Problem 5
   By Theorem 4.24, and the condition $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$
   for any $x \neq 0$, there is a point z strictly between x and 0 such that
   $f(x) = \frac{f^{(n)}(z)}{n!}(x)^n$
   Since $f^{(n)}$ is bounded, $\exists N$ such that $|f^{(n)}(x)| < N$ for all $x \in (-1,1)$
   $|f(x)| = |\frac{f^{(n)}(z)}{n!}(x)^n| \leqslant M|x|^n$ where $M = \frac{N}{n!}$.

7. 4.4 Problem 7

   – solution 1
     Let $g(h) = f(x_0+h) - 2f(x_0) + f(x_0-h)$, then
     $g'(h) = f'(x_0+h) - f'(x_0-h)$
     $g''(h) = f''(x_0+h) + f''(x_0-h)$
     $g(0) = 0$, $g'(0) = 0$
     By theorem 4.24, or Lagrange Remainder Theorem,
     for each h there is a $z = z(h) \in (0,h)$ such that
     $g(h) = \frac{g''(z)}{2!}h^2$
     $\Rightarrow f(x_0+h) - 2f(x_0) + f(x_0-h) = g(h) = \frac{g''(z)}{2!}h^2$
     $\Rightarrow \lim_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = \lim_{h\to 0} \frac{g''(z)}{2} = \lim_{h\to 0} \frac{f''(x_0+z)+f''(x_0-z)}{2} = f''(x_0)$
     The last equality is true because $z(h) \to 0$ as $h \to 0$.
   – solution 2
     Since $\lim_{h\to 0} f(x_0+h) - 2f(x_0) + f(x_0-h) = 0$ and
     $\lim_{h\to 0} h^2 = 0$,
     by L'hopital's rule,
     $\lim_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h}$
     again we have, $\lim_{h\to 0} f'(x_0+h) - f'(x_0-h) = 0$ and

$\lim_{h\to 0} 2h = 0$
By L'hopital's rule
$\Rightarrow \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h} = \lim_{h\to 0} \frac{f''(x_0+h)+f''(x_0-h)}{2} = f''(x_0)$
$\Rightarrow \lim_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = f''(x_0)$

8. 8.1 Problem 2

a. $f(x) = \int_0^x \frac{1}{1+t^2} dt$
$f'(x) = \frac{1}{1+x^2}$
$f''(x) = \frac{-2x}{(1+x^2)^2}$
$f'''(x) = \frac{-2}{(1+x^2)^2} + \frac{8x}{(1+x^2)^3}$

$$\begin{aligned} p_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= x - \frac{2}{3!}x^3 \\ &= x - \frac{1}{3}x^3 \end{aligned}$$

b. $f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$

$$\begin{aligned} p_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= x - \frac{1}{3!}x^3 \\ &= x - \frac{1}{6}x^3 \end{aligned}$$

c. $f(x) = \sin x + x^{200}$
$f'(x) = \cos x + 200x^{199}$
$f''(x) = -\sin x + (200)(199)x^{198}$
$f'''(x) = -\cos x + (200)(199)(198)x^{197}$

$$\begin{aligned} p_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= x - \frac{1}{3!}x^3 \\ &= x - \frac{1}{6}x^3 \end{aligned}$$

d. $f(x) = \sqrt{2-x}$
$f'(x) = \frac{-1}{2(2-x)^{\frac{1}{2}}}$
$f''(x) = -\frac{1}{4(2-x)^{\frac{3}{2}}}$

$f'''(x) = -\frac{3}{8(2-x)^{\frac{5}{2}}}$

$$\begin{aligned} p_3(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 - \frac{1}{2}(x-1) - \frac{1}{2!}\cdot\frac{1}{4}(x-1)^2 - \frac{1}{3!}\cdot\frac{3}{8}(x-1)^3 \\ &= 1 - \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 \end{aligned}$$

9. 8.1 Problem 4
   Since $p_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$
   we know that $f(0) = 1$, $f'(0) = 4$, $f''(0) = -2$
   Since f has three derivatives, $\Rightarrow f$, f', and f'' are continuous.
   $\exists \delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$ such that
   $|f(x) - f(0)| < \frac{1}{2}$ for $|x| < \delta_1$
   $|f'(x) - f'(0)| < \frac{1}{2}$ for $|x| < \delta_2$
   $|f''(x) - f''(0)| < \frac{1}{2}$ for $|x| < \delta_3$
   $\Rightarrow$ for $|x| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$
   $f(x) > f(0) - \frac{1}{2} = \frac{1}{2} > 0$
   $f'(x) > f'(0) - \frac{1}{2} = 4 - \frac{1}{2} > 0$
   $f''(x) < f''(0) + \frac{1}{2} = -2 + \frac{1}{2} < 0$
   Hence f is positive for $|x| < \delta$,
   $f' > 0$ for $|x| < \delta$, which implies f is strictly increasing for $|x| < \delta$
   $f'' < 0$ for $|x| < \delta$, which implies f' is strictly decreasing for $|x| < \delta$.

10. 8.2 Problem 2
    $f(x) = (1+x)^{\frac{1}{3}}$
    $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$
    $f''(x) = \frac{-2}{9}(1+x)^{-\frac{5}{3}}$
    $f'''(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$
    By the Lagrange remainder theorem, for each $x > 0$ there exists $c_x \in (0, x)$ such that
    $f(x) = f(0) + f'(0)x + \frac{f''(c_x)}{2!}x^2$
    $\frac{f''(c_x)}{2!}x^2 = -\frac{1}{9}(1+c_x)^{\frac{5}{3}}x^2 < 0$ for each $x > 0$
    $\Rightarrow f(x) < f(0) + f'(0)x = 1 + \frac{1}{3}$
    Again by the Lagrange remainder theorem, for each $x > 0$ there exists $d_x \in (0, x)$ such that
    $f(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(d_x)}{3!}x^3$
    $\frac{f'''(d_x)}{3!}x^3 = \frac{5}{81}(1+d_x)^{-\frac{8}{3}}x^3 > 0$ for each $x > 0$
    $\Rightarrow f(x) > f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{x}{3} - \frac{x^2}{9}$

    $\Rightarrow 1 + \frac{x}{3} - \frac{x^2}{9} < (1+x)^{\frac{1}{3}} < 1 + \frac{x}{3}$ for $x > 0$

11. 8.2 Problem 8
    $(\Longrightarrow)$
    Suppose that x_0 is a root of order k of the polynomial p
    then $p(x) = (x - x_0)^k r(x)$, where $r(x_0) \neq 0$
    Differentiating directly,

$p'(x) = k(x-x_0)^{k-1}r(x) + (x-x_0)^k r'(x)$
$p''(x) = k(k-1)(x-x_0)^{k-2}r(x) + 2k(x-x_0)^{k-1}r'(x) + (x-x_0)^k r''(x)$
$\vdots$
$p^{(k-1)}(x) = \sum_{i=0}^{k-1} \frac{(k-1)!}{(k-1-i)!i!}\left((x-x_0)^k\right)^{(k-1-i)} r^{(i)}(x)$
$p^{(k)}(x) = \sum_{i=0}^{k} \frac{(k)!}{(k-i)!i!}\left((x-x_0)^k\right)^{(k-i)} r^{(i)}(x) = \sum_{i=0}^{k}(\frac{k!}{(k-i)!i!})(\frac{k!}{i!})(x-x_0)^i r^{(i)}(x)$
for $p(x)$, $p'(x)$, to $p^{(k-1)}(x)$, all the terms are multiples of $(x-x_0)$,
so $p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0$
for $p^{(k)}(x)$, all terms except the term $k!r(x)$ are multiples of $(x-x_0)$.
So $p^{(k)}(x_0) = k!r(x_0) \neq 0$.

$(\Longleftarrow)$
Suppose $p(x)$ is a polynomial of degree n.
Then the nth Taylor polynomial for p at x_0 is p itself.
$p(x) = \sum_{l=0}^{n} \frac{p^{(l)}(x)}{l!}(x-x_0)^l$
Since $p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0$,
$p(x) = \sum_{l=k}^{n} \frac{p^{(l)}(x)}{l!}(x-x_0)^l = (x-x_0)^k \sum_{l=0}^{n-k} \frac{p^{(l+k)}(x)}{(l+k)!}(x-x_0)^l$
let $r(x) = \sum_{l=0}^{n-k} \frac{p^{(l+k)}(x)}{(l+k)!}(x-x_0)^l$
then we know that $r(x_0) = p^{(k)}(x_0) \neq 0$
Therefore we have $p(x) = (x-x_0)^k r(x)$, where $r(x_0) \neq 0$
So x_0 is a root of p with order k.

12. 8.2 Problem 11

a. Since $f^{(n+1)}(x)$ is continuous and $f^{(n+1)}(x_0) > 0$,
there exists $\delta > 0$ such that $f^{(n+1)}(x) > 0$ for x in $|x_0 - x| < \delta$.
By the Lagrange remainder theorem, for each $x \neq$ with $|x - x_0| < \delta$ there is a c_x strictly between x_0 and x satisfying
$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
Since $f^{(k)}(x) = 0$ for $1 \leqslant k \leqslant n$,
$f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
For $|x - x_0| < \delta$, we have $|x_0 - c_x| < |x_0 - x| < \delta$, so we have $f^{(n+1)}(c_x) > 0$,
also $n+1$ is even gives $(x-x_0)^{n+1} > 0$
$\Rightarrow \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} > 0$
$\Rightarrow f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} > f(x_0) \Rightarrow x_0$ is a local minimizer.

b. Since $f^{(n+1)}(x)$ is continuous and $f^{(n+1)}(x_0) < 0$,
there exists $\delta > 0$ such that $f^{(n+1)}(x) < 0$ for x in $|x_0 - x| < \delta$.
By the Lagrange remainder theorem, for each $x \neq 0$ there is a c_x strictly between x_0 and x satisfying
$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
Since $f^{(k)}(x) = 0$ for $1 \leqslant k \leqslant n$,
$f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
For $|x - x_0| < \delta$, we have $|x_0 - c_x| < |x_0 - x| < \delta$, so we have $f^{(n+1)}(c_x) < 0$,
also $n+1$ is even gives $(x-x_0)^{n+1} > 0$
$\Rightarrow \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} < 0$

$\Rightarrow f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} < f(x_0) \Rightarrow x_0$ is a local maximizer.

c. Suppose $f^{(n+1)}(x_0) > 0$,
since $f^{(n+1)}(x)$ is continuous,
there exists $\delta > 0$ such that $f^{(n+1)}(x) > 0$ for x in $|x_0 - x| < \delta$.
By the Lagrange remainder theorem, for each $x \neq 0$ there is a c_x strictly between x_0 and x satisfying
$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
Since $f^{(k)}(x) = 0$ for $1 \leqslant k \leqslant n$,
$f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$
For $|x - x_0| < \delta$, we have $|x_0 - c_x| < |x_0 - x| < \delta$, so we have $f^{(n+1)}(c_x) > 0$
Since $n+1$ is odd, for $x > x_0$,
$f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} > f(x_0)$
for $x < x_0$,
$f(x) = f(x_0) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} < f(x_0)$
$\Rightarrow x_0$ is not local minimizer nor a local maximizer.
The case where $f^{(n+1)}(x_0) < 0$ is similar.

13. 8.2 Problem 12

a. By the Lagrange remainder theorem, for each $x \neq 0$, there exists c_h strictly between x_0 and $x_0 + h$, such that
$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(c_h)}{2!}h^2$
Since $f'''(x) > 0$, $f''(x)$ is strictly increasing and is one-to-one, so c_h is unique.
let $\theta(h) = \frac{c_h - x_0}{h}$, clearly, $0 < \theta(h) < 1$, and
$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0 + \theta(h)h)}{2!}h^2$

b. By the Lagrange remainder theorem, for each $h \neq 0$ there exists d_h strictly between x_0 and $x_0 + h$ such that
$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(d_h)}{3!}h^3$,
So with the equation from (a.), we have
$\frac{f''(x_0+\theta(h)h)}{2!}h^2 = \frac{f''(x_0)}{2!}h^2 + \frac{f'''(d_h)}{3!}h^3$
Since $h \neq 0$
$\Rightarrow \frac{\frac{f''(x_0+\theta(h)h)}{2!} - \frac{f''(x_0)}{2!}}{h} = \frac{f'''(d_h)}{3!}$
$\Rightarrow \lim_{h\to 0}\left(\theta(h)\frac{f''(x_0+\theta(h)h) - f''(x_0)}{\theta(h)h}\right) = \lim_{h\to 0}\frac{f'''(d_h)}{3} = \frac{f'''(x_0)}{3} > 0$
$\Rightarrow \left(\lim_{h\to 0}\theta(h)\right)\left(\lim_{h\to 0}\frac{f''(x_0+\theta(h)h) - f''(x_0)}{\theta(h)h}\right) = \left(\lim_{h\to 0}\theta(h)\right)f'''(x_0) = \frac{f'''(x_0)}{3} > 0$
$\Rightarrow \lim_{h\to 0}\theta(h) = \frac{1}{3}$

14. 8.3 Problem 1

a. $f(x) = \sin x$
$f'(x) = \cos x$
$\vdots$

$|f^{(n)}(x)| \leqslant 1$ for all n, and all x
By theorem 8.14,
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^{(k)}}{(2k+1)!} x^{2k+1}$
So we know that for every x, the Taylor series converges.

b. $f(x) = \cos x$
$f'(x) = -\sin x$
$\vdots$

$|f^{(n)}(x)| \leqslant 1$ for all n, and all x
By theorem 8.14,
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^{(k+1)}}{(2k)!} (x - \pi)^{2k}$
So we know that for every x, the Taylor series converges.