HW 9

1.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for} \quad x \neq 0 \\ 0 \quad \text{for} & x = 0 \end{cases}$$

Show that f is infinitely differentiable.

(i)

It is clear that f is infinitely differentiable for $x \neq 0$.
We will show that $f^{(k)}(x) = p_k(\frac{1}{x})e^{-\frac{1}{x^2}}$ for all $x \neq 0$, where $p_k(x)$ is a polynomial.
We show this by induction.
$f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$, so $p_1(x) = -2x^3$ is a polynomial
Suppose $f^{(k)}(x) = p_k(\frac{1}{x})e^{-\frac{1}{x^2}}$, where $p_k(x)$ is a polynomial.
Then $f^{(k+1)}(x) = p_k'(\frac{1}{x})(\frac{-1}{x^2})e^{-\frac{1}{x^2}} + p_k(\frac{1}{x})(\frac{2}{x^3})e^{-\frac{1}{x^2}}$
$p_{k+1}(\frac{1}{x}) = p_k'(\frac{1}{x})(\frac{-1}{x^2}) + p_k(\frac{1}{x})(\frac{2}{x^3})$
We can see that $p_{k+1}(x)$ is a polynomial.

(ii)

Next we show that $\lim_{x\to 0}\left(\frac{1}{x^m}e^{-\frac{1}{x^2}}\right) = 0$ for all $m \in \mathbb{N}$
$\lim_{x\to 0}\left(\frac{1}{x^m}e^{-\frac{1}{x^2}}\right) = \lim_{y\to\infty} y^m e^{-y^2} = \lim_{y\to\infty} y^m \frac{1}{e^{y^2}} \overset{l'hopital}{=} \lim_{y\to\infty} \frac{my^{m-1}}{2ye^{y^2}}$
$= \lim_{y\to\infty} \frac{my^{m-2}}{2e^{y^2}} = \ldots = \lim_{y\to\infty} \frac{c}{e^{y^2}}$ or $\lim_{y\to\infty} \frac{c}{ye^{y^2}} = 0$. ($c$ here represent different constants.)

(iii)

Last we show that $f^{(k)}(0)$ exists and equals 0 for all k.
We also prove this by induction.
First, we know that
$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{1}{x}e^{-\frac{1}{x^2}} = 0$
Suppose $f^{(k)}(0) = 0$
Then
$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = \lim_{x\to 0} \frac{f^{(k)}(x)}{x} \overset{by(i)}{=} \lim_{x\to 0}(\frac{1}{x}p_k(\frac{1}{x})e^{-\frac{1}{x^2}}) \overset{by(ii)}{=} 0$
So f is infinitely differentiable at 0.

2. Compute the Taylor expansion of $\cos x$ at $x = 0$.

$f(x) = \cos x$
$f'(x) = -\sin x$
$\vdots$

$|f^{(n)}(x)| \leqslant 1$ for all n, and all $x \in \mathbb{R}$
By theorem 8.14,
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x)^{2k}$

3. Compute the Taylor expansion of $\sinh x$ at $x = 0$.

$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$
$f'(x) = \cosh x = \frac{e^x + e^{-x}}{2}$
$\vdots$

$|f^{(n)}(x)| \leqslant e^{|x|}$ for all n and all $x \in \mathbb{R}$
for $x < R$, $|f^{(n)}(x)| \leqslant e^R$
By theorem 8.14,
$\sinh x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k$ for $x < R$
Since the above is true for arbitrary R,
$\sinh x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ for all $x \in \mathbb{R}$