HW 10

1. Let $f : [a,b] \to \mathbb{R}$ be bounded. Show that if $L(f,P) \leqslant A \leqslant U(f,P)$ for all partitions P of $[a,b]$. Then, $A \in [\overline{L}(f), \underline{U}(f)]$.

$$A \leqslant U(f,P) \Rightarrow A \leqslant \inf_P U(f,P) = \underline{U}(f)$$

$$A \geqslant L(f,P) \Rightarrow A \geqslant \sup_P L(f,P) = \overline{L}(f)$$

$$A \in [\overline{L}(f), \underline{U}(f)]$$

2. Let $f : [a,b] \to \mathbb{R}$ be bounded. Prove that f is Riemann integrable iff there exists a sequence of partitions that is Archimedean for f.

($\Longrightarrow$)
f is Riemann integrable $\Rightarrow \overline{L}(f) = \underline{U}(f)$
$\overline{L}(f) = \sup\{L(f,P) : P$ is a partition over $[a,b]\}$
$\Rightarrow \exists$ partition P_1^n of $[a,b]$ such that $L(f,P_1^n) > \overline{L}(f) - \frac{1}{n}$
$\underline{U}(f) = \inf\{L(f,P) : P$ is a partition over $[a,b]\}$
$\Rightarrow \exists$ partition P_2^n of $[a,b]$ such that $L(f,P_2^n) < \underline{U}(f) + \frac{1}{n}$
Let P^n be a common refinement of P_1^n and P_2^n
Then $L(f,P_1^n) \leqslant L(f,P^n) \leqslant U(f,P^n) \leqslant U(f,P_2^n)$
$U(f,P^n) - L(f,P^n) \leqslant U(f,P_2^n) - L(f,P_1^n) < \underline{U}(f) + \frac{1}{n} - \overline{L}(f) + \frac{1}{n} = \frac{2}{n}$
Therefore, we can find a sequence of partitions $\{P^n\}$ such that
$\lim_{n\to\infty}\{U(f,P^n) - L(f,P^n)\} = 0$
$\{P^n\}$ is Archimedean.

($\Longleftarrow$)
Suppose there is an Archimedean sequence of partitions $\{P^n\}$
$\lim_{n\to\infty}\{U(f,P^n) - L(f,P^n)\} = 0$
By the definition of inf and sup
$0 < \underline{U}(f) - \overline{L}(f) \leqslant U(f,P^n) - L(f,P^n)$ for all n.
Since $U(f,P^n) - L(f,P^n)$ can be arbitrarily small,
$\underline{U}(f) = \overline{L}(f) \Rightarrow f$ is Riemann integrable.

3. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to$ be integrable over $[a,b]$. Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also integrable over $[a,b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.

First show that $\alpha \int_a^b f = \int_a^b \alpha f$
f Riemann integrable $\overset{by(2)}{\Longrightarrow} \exists P^n$ such that
$\lim_{n\to\infty} U(f,P^n) - L(f,P^n) = 0$
By the definition of $U(f,P)$ and $L(f,P)$, we can observe that
$U(\alpha f,P) = \alpha U(f,P)$ and $L(\alpha f,P) = \alpha L(\alpha f,P)$ for $\alpha \geqslant 0$
$U(\alpha f,P) = \alpha L(f,P)$ and $L(\alpha f,P) = \alpha U(\alpha f,P)$ for $\alpha < 0$
In either case we have
$\lim_{n\to\infty} U(\alpha f,P^n) - L(\alpha f,P^n) = 0$
moreover, $\overline{L}(\alpha f) = \underline{U}(\alpha) = \alpha \underline{U}(f) = \alpha \overline{L}(f)$
So $\alpha \int_a^b f = \int_a^b \alpha f$.

Then we prove that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$
There exists an Archimedean sequence of partitions $\{P_f^n\}$ for f
There exists an Archimedean sequence of partitions $\{P_g^n\}$ for g
Let $\{P^n\}$ be a common refinement of $\{P_f^n\}$ and $\{P_g^n\}$
Then $\{P^n\}$ is an Archimedean sequence of partitions for f and g.
$\lim_{n\to\infty}\{U(f,P^n) - L(f,P^n)\} = 0$
$\lim_{n\to\infty}\{U(g,P^n) - L(g,P^n)\} = 0$
For all n,
$L(f,P^n) + L(g,P^n) \leqslant L(f+g,P^n) \leqslant U(f+g,P^n) \leqslant U(f,P^n) + U(g,P^n)$
Taking the limit on all terms, since both ends are equal, we get
$\lim_{n\to\infty} L(f,P^n) + \lim_{n\to\infty} L(g,P^n) = \lim_{n\to\infty} L(f+g,P^n) = \lim_{n\to\infty} U(f+g,P^n) = \lim_{n\to\infty} U(f,P^n) + \lim_{n\to\infty} U(g,P^n)$
So $\{P^n\}$ is Archimedean for $f+g$, and by problem 2 we know that (f+g) is integrable.
Moreover, by the above equality, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Combining both parts, we get linearity
$\int_a^b (\alpha f + \beta g) = \int_a^b \alpha f + \int_a^b \beta g = \alpha \int_a^b f + \beta \int_a^b g$

4. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable over $[a,b]$. Moreover, suppose $f(x) \leqslant g(x) \ \ \forall x \in [a,b]$, then $\int_a^b f \leqslant \int_a^b g$.

   Since $f(x) - g(x) \geqslant 0$
   $\int_a^b (f-g) \geqslant L(f-g,P) \geqslant 0$ where P is any partition over $[a,b]$
   By linearity, $\int_a^b f - \int_a^b g = \int_a^b (f-g)$
   $\Rightarrow \int_a^b f - \int_a^b g \geqslant 0$
   $\Rightarrow \int_a^b f \geqslant \int_a^b g$