HW 11

1. Prove that sets of measure zero contain no intervals.

   A has measure zero $\Rightarrow \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \cup_{i=1}^{\infty}(a_i, b_i)\} = 0$ where inf is taken over all collections of countable number of intervals which their union contains A.
   Suppose there is some interval $(c, d) \subset A$
   Then for any collection of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ containing A,
   $\cup_{i=1}^{\infty}(a_i, b_i) \supset A \supset (c, d)$
   $\sum_{i=1}^{\infty}(b_i - a_i) \geqslant (d - c) > 0$ for all collection of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ containing A, then measure of A would be greater than or equal to $d - c > 0$ which is a contradiction to A has measure zero.
   Therefore A contains no open interval, but this also implies that A cannot contain any interval.

2. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and nonnegative, with f positive over $(c, d) \subset [a, b]$. Then $\int_a^b f dx > 0$.

   By the Lebesgue theorem, f must be continuous almost everywhere (discontinuous at a set of measure zero).
   Since by the previous problem, sets of measure zero contain no intervals,
   $\exists p \in (c, d)$ such that f is continuous at p.
   We have $f(p) > 0$.
   By continuity of f, $\exists\delta$, $0 < \delta < \min\{b - p, p - a\}$, such that if $|x - p| \leqslant \delta$, then $|f(x) - f(p)| < \frac{f(p)}{2}$,
   which implies that if $|x - p| \leqslant \delta$, then $f(x) > \frac{f(p)}{2}$
   $\int_{p-\delta}^{p+\delta} f \geqslant \int_{p-\delta}^{p+\delta} \frac{f(p)}{2} = \delta f(p) > 0$ where we use the monotonicity property of the integral.
   Since $f(x) \geqslant 0$ for all $x \in [a, b]$, again by monotonicity property, $\int_a^{p-\delta} f \geqslant 0$ and $\int_{p+\delta}^b f \geqslant 0$
   combining the three parts,
   $\int_a^b f = \int_a^{p-\delta} f + \int_{p-\delta}^{p+\delta} f + \int_{p+\delta}^b f \geqslant \delta f(p) > 0$

3. Let $G : [a, b] \to \mathbb{R}$ be nondecreasing and continuous over $[a, b]$ and differentiable over (a, b). Suppose g is an extension of G' to $[a, b]$ that is Riemann integrable. Let $f : [G(a), G(b)] \to \mathbb{R}$ be Riemann integrable and have a primitive F, where $F : [G(a), G(b)] \to \mathbb{R}$ is continuous over $[G(a), G(b)]$ and differentiable over $(G(a), G(b))$. Then

$$\int_{G(a)}^{G(b)} f(y)dy = \int_a^b f(G(x))g(x)dx$$

   Let $P = [p_0, \cdots, p_l]$ be the partition of $[G(a), G(b)]$ associated with the primitive F.
   Let $Q = [q_0, \cdots, q_m]$ be the partition of $[a, b]$ associated with the primitive G.
   Since G is not strictly increasing, the inverse of G may not be well defined.
   Still, we can choose any of the points that maps to $p_1, \cdots, p_{l-1}$ to be $G^{-1}(p_1), \cdots, G^{-1}(p_{l-1})$, and let $G^{-1}(p_0) = a$ and $G^{-1}(p_l) = b$.
   Then since G is nondecreasing, $G^{-1}(p_0) \leqslant G^{-1}(p_1) \leqslant \cdots \leqslant G^{-1}(p_l)$.
   Let $G^{-1}(P) = [G^{-1}(p_0), \cdots, G^{-1}(p_l)]$ and $R = Q \vee G^{-1}(P) = [r_0, \cdots, r_n]$.
   Then we have that $F(G)$ is differentiable over (r_{i-1}, r_i) $(i = 1, \cdots, n)$.

By the Chain Rule we have
$F(G)'(x) = F'(G(x))G'(x) = f(G(x))g(x)$ for every $x \in (r_{i-1}, r_i)$
So $F(G)$ is a primitive of $f(G)g$ over $[a, b]$
The rest of the proof is the same as the increasing case. (Proposition 3.1 in the notes).

4. Find an example such that $\int_{G(a)}^{G(b)} f(y)dy = \int_a^b f(G(x))g(x)dx$ fails.

Let

$$f(x) = \begin{cases} x & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0 \end{cases}$$

and $G(x) = x^2$ then $g(x) = 2x$
Let $a = -1$ and $b = 1$
$\int_{G(a)}^{G(b)} f(y)dy = \int_1^1 f(x)dx = 0 \neq \frac{1}{2} = \int_0^1 2x^3 dx + \int_{-1}^0 0dx = \int_a^b f(G(x))g(x)dx$