Advanced Calculus: MATH 410
Riemann Integrals and Integrability

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1. Definite Integrals

In this section we revisit the definite integral that you were introduced to when you first studied calculus. You undoubtedly learned that given a positive function f over an interval $[a, b]$ the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

provided it was defined, was a number equal to the area under the graph of f over $[a, b]$. You also likely learned that the definite integral was defined as a limit of Riemann sums. The Riemann sums you most likely used involved partitioning $[a, b]$ into n uniform subintervals of length $(b - a)/n$ and evaluating f at either the right-hand endpoint, the left-hand endpoint, or the midpoint of each subinterval. At the time your understanding of the notion of limit was likely more intuitive than rigorous. In this section we present the *Riemann Integral*, a rigorous development of the definite integral built upon the rigorous understanding of limit that you have studied earlier in this course.

1.1. **Partitions and Darboux Sums.**

We will consider very general partitions of the interval $[a, b]$, not just those with uniform subintervals.

Definition 1.1. *Let* $[a, b] \subset \mathbb{R}$*. A* partition *of the interval* $[a, b]$ *is specified by* $n \in \mathbb{N}$*, and* $\{x_i\}_{i=1}^n \subset [a, b]$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\,.$$

The partition P *associated with these points is defined to be the ordered collection of* n *subintervals of* $[a, b]$ *given by*

$$P = \big([x_{i-1}, x_i] \,:\, i = 1, \cdots, n\big)$$

This partition is denoted $P = [x_0, x_1, \cdots, x_{n-1}, x_n]$*. Each* x_i *for* $i = 0, \cdots, n$ *is called a* partition point *of* P*, and for each* $i = 1, \cdots, n$ *the interval* $[x_{i-1}, x_i]$ *is called a* i^{th} subinterval in P*. The* partition thickness *or* gap*, denoted* $|P|$*, is defined by*

$$|P| = \max\big\{x_i - x_{i-1} \,:\, i = 1, \cdots, n\big\}\,.$$

The approach to the definite integral taken here is not based on Riemann sums, but rather on Darboux sums. This is because Darboux sums are well-suited for analysis by the tools we have developed to establish the existence of limits. We will be able to recover results about Riemann sums because, as we will show, every Riemann sum is bounded by two Darboux sums.

Let $f : [a, b] \to \mathbb{R}$ be bounded. Set

$$\underline{m} = \inf\left\{ f(x) \,:\, x \in [a, b] \right\}, \qquad \overline{m} = \sup\left\{ f(x) \,:\, x \in [a, b] \right\}. \tag{1}$$

Because f is bounded, one knows that $-\infty < \underline{m} \le \overline{m} < \infty$.

Let $P = [x_0, \cdots, x_n]$ be a partition of $[a, b]$. For each $i = 1, \cdots, n$ set

$$\underline{m}_i = \inf\left\{ f(x) \,:\, x \in [x_{i-1}, x_i] \right\},$$
$$\overline{m}_i = \sup\left\{ f(x) \,:\, x \in [x_{i-1}, x_i] \right\}.$$

Clearly $\underline{m} \le \underline{m}_i \le \overline{m}_i \le \overline{m}$ for every $i = 1, \cdots, n$.

Definition 1.2. *The* lower and upper Darboux sums *associated with the function f and partition P are respectively defined by*

$$L(f, P) = \sum_{i=1}^{n} \underline{m}_i \,(x_i - x_{i-1}), \quad U(f, P) = \sum_{i=1}^{n} \overline{m}_i \,(x_i - x_{i-1}). \tag{2}$$

Clearly, the Darboux sums satisfy the bounds

$$\underline{m}\,(b - a) \le L(f, P) \le U(f, P) \le \overline{m}\,(b - a). \tag{3}$$

These inequalities will all be equalities when f is a constant.

Remark. A Riemann sum associated with the partition P is specified by selecting a quadrature point $q_i \in [x_{i-1}, x_i]$ for each $i = 1, \cdots, n$. Let $Q = (q_1, \cdots, q_n)$ be the n-tuple of quadrature points. The associated Riemann sum is then

$$R(f, P, Q) = \sum_{i=1}^{n} f(q_i)\,(x_i - x_{i-1}).$$

It is easy to see that for any choice of quadrature points Q one has the bounds

$$L(f, P) \le R(f, P, Q) \le U(f, P). \tag{4}$$

Moreover, one can show that

$$\begin{aligned} L(f, P) &= \inf\left\{ R(f, P, Q) \,:\, Q \text{ are quadrature points for } P \right\}, \\ U(f, P) &= \sup\left\{ R(f, P, Q) \,:\, Q \text{ are quadrature points for } P \right\}. \end{aligned} \tag{5}$$

The bounds (4) are thereby sharp.

Exercise. Prove (5)

1.2. **Refinements.** We now introduce the notion of a refinement of a partition.

Definition 1.3. *Given a partition P of an interval $[a,b]$, a partition P^* of $[a,b]$ is called a* refinement *of P provided every partition point of P is a partition point of P^*.*

If $P = [x_0, x_1, \cdots, x_{n-1}, x_n]$ and P^* is a refinement of P then P^* induces a partition of each $[x_{i-1}, x_i]$, which we denote by P_i^*. For example, if $P^* = [x_0^*, x_1^*, \cdots, x_{n^*-1}^*, x_{n^*}^*]$ with $x_{j_i}^* = x_i$ for each $i = 0, \cdots, n$ then $P_i^* = [x_{j_{i-1}}^*, \cdots, x_{j_i}^*]$. Observe that

$$L(f, P^*) = \sum_{i=1}^{n} L(f, P_i^*) \,, \qquad U(f, P^*) = \sum_{i=1}^{n} U(f, P_i^*) \,. \tag{6}$$

Moreover, upon applying the bounds (3) to P_i^* for each $i = 1, \cdots, n$, we obtain the bounds

$$\underline{m}_i \,(x_i - x_{i-1}) \leq L(f, P_i^*) \leq U(f, P_i^*) \leq \overline{m}_i \,(x_i - x_{i-1}) \,. \tag{7}$$

This observation is key to the proof of the following.

Lemma 1.1. (Refinement) *Let $f : [a,b] \to \mathbb{R}$ be bounded. Let P be a partition of $[a,b]$ and P^* be a refinement of P. Then*

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P) \,. \tag{8}$$

Proof. It follows from (2), (7), and (6) that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} \underline{m}_i \,(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} L(f, P_i^*) = L(f, P^*) \\ &\leq U(f, P^*) = \sum_{i=1}^{n} U(f, P_i^*) \\ &\leq \sum_{i=1}^{n} \overline{m}_i \,(x_i - x_{i-1}) = U(f, P) \,. \end{aligned}$$

□

1.3. **Comparisons.** A key step in our development will be to develop comparisons of $L(f,P^1)$ and $U(f,P^2)$ for any two partitions P^1 and P^2, of $[a,b]$.

Definition 1.4. *Given any two partitions, P^1 and P^2, of $[a,b]$ we define $P^1 \vee P^2$ to be the partition whose set of partition points is the union of the partition points of P^1 and the partition points of P^2. We call $P^1 \vee P^2$ the* supremum *of P^1 and P^2.*

It is easy to argue that $P^1 \vee P^2$ is the smallest partition of $[a,b]$ that is a refinement of both P^1 and P^2. It is therefore sometimes called the *smallest common refinement* of P^1 and P^2.

Lemma 1.2. (Comparison) *Let $f:[a,b]\to\mathbb{R}$ be bounded. Let P^1 and P^2 be partitions of $[a,b]$. Then*

$$L(f,P^1) \le U(f,P^2)\,. \tag{9}$$

Proof. Because $P^1 \vee P^2$ is a refinement of both P^1 and P^2, it follows from the Refinement Lemma that

$$L(f,P^1) \le L(f,P^1\vee P^2) \le U(f,P^1\vee P^2) \le U(f,P^2)\,.$$

□

Because the partitions P^1 and P^2 on either side of inequality (9) are independent, we may obtain sharper bounds by taking the supremum over P^1 on the left-hand side, or the infimum over P^2 on the right-hand side. Indeed, we prove the following.

Lemma 1.3. (Sharp Comparison) *Let $f:[a,b]\to\mathbb{R}$ be bounded. Let*

$$\begin{aligned} \overline{L}(f) &= \sup\big\{L(f,P)\,:\, P \text{ is a partition of } [a,b]\big\}\,,\\ \underline{U}(f) &= \inf\big\{U(f,P)\,:\, P \text{ is a partition of } [a,b]\big\}\,. \end{aligned} \tag{10}$$

Let P^1 and P^2 be partitions of $[a,b]$. Then

$$L(f,P^1) \le \overline{L}(f) \le \underline{U}(f) \le U(f,P^2)\,. \tag{11}$$

Moreover, if

$$L(f,P) \le A \le U(f,P) \quad \text{for every partition } P \text{ of } [a,b]\,,$$

then $A \in [\overline{L}(f), \underline{U}(f)]$.

Remark. Because it is clear from (10) that $\overline{L}(f)$ and $\underline{U}(f)$ depend on $[a,b]$, strictly speaking these quantities should be denoted $\overline{L}(f,[a,b])$ and $\underline{U}(f,[a,b])$. This would be necessary if more than one interval was involved in the discussion. However, that is not the case here. We therefore embrace the less cluttered notation.

Proof. If we take the infimum of the right-hand side of (9) over P^2, we obtain

$$L(f, P_1) \le \underline{U}(f)\,.$$

If we then take the supremum of the left-hand side above over P^1, we obtain

$$\overline{L}(f) \le \underline{U}(f)\,.$$

The bound (11) then follows.

The proof of the last assertion is left as an exercise. □

Exercise. Prove the last assertion of Lemma 1.3.

1.4. **Definition of the Riemann Integral.** We are now ready to define the definite integral of Riemann.

Definition 1.5. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is said to be* Riemann integrable *over $[a, b]$ whenever $\overline{L}(f) = \underline{U}(f)$. In this case we call this common value the* Riemann integral *of f over $[a, b]$ and denote it by $\int_a^b f$:*

$$\int_a^b f = \overline{L}(f) = \underline{U}(f)\,. \tag{12}$$

Then f is called the integrand *of the integral, a is called the* lower endpoint *(or* lower limit*) of integration, while b is called the* upper endpoint *(or* upper limit*) of integration.*

Remark. We will call a and b the endpoints of integration rather than the limits of integration. The word "limit" does enough work in this subject. We do not need to adopt terminology that can lead to confusion.

We begin with the following characterizations of integrability.

Theorem 1.1. (Riemann-Darboux) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then the following are equivalent:*

(1) *f is Riemann integrable over $[a, b]$ (i.e. $\overline{L}(f) = \underline{U}(f)$);*
(2) *for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that*

$$0 \le U(f, P) - L(f, P) < \epsilon\,;$$

(3) *there exists a unique $A \in \mathbb{R}$ such that*

$$L(f, P) \le A \le U(f, P) \quad \textit{for every partition } P \textit{ of } [a, b]\,. \tag{13}$$

Moreover, in case (3) $A = \int_a^b f$.

Remark. The Sharp Comparison Lemma shows that (13) holds if and only if $A \in [\overline{L}(f), \underline{U}(f)]$. The key thing to be established when using characterization (3) is therefore the uniqueness of such an A.

Proof. First we show that (1) $\implies$ (2). Let $\epsilon > 0$. By the definition (10) of $\overline{L}(f)$ and $\underline{U}(f)$, we can find partitions P^L and P^U of $[a, b]$ such that

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f, P^L) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f, P^U) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Let $P = P^L \vee P^U$. Because the Comparison Lemma implies that $L(f, P^L) \leq L(f, P)$ and $U(f, P) \leq U(f, P^U)$, it follows from the above inequalities that

$$\overline{L}(f) - \frac{\epsilon}{2} < L(f, P) \leq \overline{L}(f)\,,$$
$$\underline{U}(f) \leq U(f, P) < \underline{U}(f) + \frac{\epsilon}{2}\,.$$

Hence, if $\overline{L}(f) = \underline{U}(f)$ one concludes that

$$0 \leq U(f, P) - L(f, P) < \left(\underline{U}(f) + \frac{\epsilon}{2}\right) - \left(\overline{L}(f) - \frac{\epsilon}{2}\right) = \epsilon\,.$$

This shows that (1) $\implies$ (2).

Next we show that (2) $\implies$ (3). Suppose that (3) is false. The Sharp Comparison Lemma shows that (13) holds for every $A \in [\overline{L}(f), \underline{U}(f)]$, and that this interval is nonempty. So the only way (3) can be false is if uniqueness fails. In that case there exists A_1 and A_2 such that

$$L(f, P) \leq A_1 < A_2 \leq U(f, P) \quad \text{for every partition } P \text{ of } [a, b]\,.$$

One thereby has that

$$U(f, P) - L(f, P) \geq A_2 - A_1 > 0 \quad \text{for every partition } P \text{ of } [a, b]\,.$$

Hence, (2) must be false. It follows that (2) $\implies$ (3).

Finally, we show that (3) $\implies$ (1) and that (3) implies $A = \int_a^b f$. The Sharp Comparison Lemma shows that (13) holds if and only if $A \in [\overline{L}(f), \underline{U}(f)]$. But (3) states that such an A is unique. Hence, $A = \overline{L}(f) = \underline{U}(f)$, which implies (1) and $A = \int_a^b f$. □

Remark. Characterizations (2) and (3) of the Integrability Theorem provides a very useful criterion for establishing the integrability of a function f.

1.5. **Convergence of Riemann and Darboux Sums.** We now make a connection with the notion of a definite integral as the limit of a sequence of Riemann sums.

Recall for any given $f : [a, b] \to \mathbb{R}$ a Riemann sum associated with a partition $P = [x_0, x_1 \cdots , x_n]$ of $[a, b]$ is specified by selecting a quadrature point $q_i \in [x_{i-1}, x_i]$ for each $i = 1, \cdots , n$. Let $Q = (q_1, \cdots , q_n)$ be the n-tuple of quadrature points. The associated Riemann sum is then

$$R(f, P, Q) = \sum_{i=1}^{n} f(q_i)\,(x_i - x_{i-1})\,. \tag{14}$$

If $f : [a, b] \to \mathbb{R}$ is bounded (so that the Darboux sums $L(f, P)$ and $U(f, P)$ are defined) then for any choice of quadrature points Q one has the bounds

$$L(f, P) \leq R(f, P, Q) \leq U(f, P)\,. \tag{15}$$

A sequence of Riemann sums for any given $f : [a, b] \to \mathbb{R}$ is therefore specified by a sequence $\{P^n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ and a sequence $\{Q^n\}_{n=1}^{\infty}$ of associated quadrature points. The sequence of partitions cannot be arbitrary.

Definition 1.6. *Let $f : [a, b] \to \mathbb{R}$ be bounded. A sequence $\{P^n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ is said to be* Archimedean *for f provided*

$$\lim_{n\to\infty} \big(U(f, P^n) - L(f, P^n)\big) = 0\,. \tag{16}$$

Our main theorem is the following.

Theorem 1.2. (Archimedes-Riemann) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable over $[a, b]$ if and only if there exists a sequence of partitions of $[a, b]$ that is Archimedean for f. If $\{P^n\}_{n=1}^{\infty}$ is any such sequence then*

$$\lim_{n\to\infty} L(f, P^n) = \int_a^b f\,, \quad \textit{and} \quad \lim_{n\to\infty} U(f, P^n) = \int_a^b f\,. \tag{17}$$

Moreover, if for each partition P^n there is an associated quadrature set Q^n then

$$\lim_{n\to\infty} R(f, P^n, Q^n) = \int_a^b f\,, \tag{18}$$

where the Riemann sums $R(f, P, Q)$ are defined by (14).

Remark. The content of this theorem is that once one has found a sequence of partitions P^n such that (16) holds, then the integral $\int_a^b f$ exists and may be evaluated as the limit of any associated sequence of Darboux sums (17) or Riemann sums (18). This theorem thereby splits the task of evaluating a definite integrals into two steps. The first step is by far the easier. It is a rare integrand f for which one can find a sequence of Darboux or Riemann sums that allows one of the limits (17) or (18) to be evaluated directly.

Proof. If f is Riemann integrable over $[a, b]$ then one can use characterization (2) of the Integrability Theorem to construct a sequence of partitions that satisfies (16), and is thereby Archimedean for f. Conversely, given a sequence of partitions of $[a, b]$ that is Archimedean for f, the that f is integrable over $[a, b]$ follows directly from characterization (2) of the Integrability Theorem. The details of these arguments are left as an exercise.

Now let $\{P^n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ that is Archimedean for f and $\{Q^n\}_{n=1}^{\infty}$ be a sequence of associated quadrature points. The bounds on Riemann sums given by (15) yield the inequalities

$$L(f, P^n) \leq R(f, P^n, Q^n) \leq U(f, P^n)\,,$$

while, because f is Riemann integrable, we also have the inequalities

$$L(f, P^n) \leq \int_a^b f \leq U(f, P^n)\,.$$

It follows from these inequalities that

$$\begin{aligned} L(f, P^n) - U(f, P^n) &\leq L(f, P^n) - \int_a^b f \\ &\leq R(f, P^n, Q^n) - \int_a^b f \\ &\leq U(f, P^n) - \int_a^b f \leq U(f, P^n) - L(f, P^n)\,, \end{aligned}$$

which implies that

$$\begin{aligned} \left| L(f, P^n) - \int_a^b f \right| &\leq U(f, P^n) - L(f, P^n)\,, \\ \left| R(f, P^n, Q^n) - \int_a^b f \right| &\leq U(f, P^n) - L(f, P^n)\,, \\ \left| U(f, P^n) - \int_a^b f \right| &\leq U(f, P^n) - L(f, P^n)\,. \end{aligned}$$

Because $\{P^n\}_{n=1}^{\infty}$ is Archimedean for f, it satisfies (16), whereby the right-hand sides above vanish as n tends to ∞. The limits (17) and (18) follow. $\square$

1.6. **Partitions Theorem.** We now prove a theorem that will subsequently provide us with a simple criterion for a sequence of partitions to be Archimedean for any Riemann integrable function.

Theorem 1.3. (Partitions) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f Riemann integrable over $[a, b]$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition P of $[a, b]$ one has*

$$|P| < \delta \quad \Longrightarrow \quad 0 \leq U(f, P) - L(f, P) < \epsilon\,. \tag{19}$$

Proof. ($\Longleftarrow$) This follow immediately from the Riemann-Darboux Theorem.

($\Longrightarrow$) Let $\epsilon > 0$. Because f is Riemann integrable over $[a, b]$ the Riemann-Darboux Theorem implies that there exists a partition P^ϵ of $[a, b]$ such that

$$0 \leq U(f, P^\epsilon) - L(f, P^\epsilon) < \frac{\epsilon}{3}\,.$$

Let n^ϵ be the number of subintervals in P^ϵ. Pick $\delta > 0$ such that

$$n^\epsilon 2M\delta < \frac{\epsilon}{3}\,, \quad \text{where} \quad M = \sup\big\{|f(x)| \,:\, x \in [a, b]\big\}\,.$$

We must show that (19) holds for this δ.

Now let $P = [x_0, x_1, \cdots, x_n]$ be an arbitrary partition of $[a, b]$ such that $|P| < \delta$. Set $P^* = P \vee P^\epsilon$. We consider

$$\begin{aligned} 0 \leq U(f, P) - L(f, P) = {} & \big(U(f, P) - U(f, P^*)\big) \\ & + \big(U(f, P^*) - L(f, P^*)\big) \\ & + \big(L(f, P^*) - L(f, P)\big)\,. \end{aligned} \tag{20}$$

We will prove the theorem by showing that each of the three terms in parentheses on the right-hand side above is less than $\epsilon/3$.

Because P^* is a refinement of P^ϵ, the Refinement Lemma implies that

$$0 \leq U(f, P^*) - L(f, P^*) \leq U(f, P^\epsilon) - L(f, P^\epsilon) < \frac{\epsilon}{3}\,.$$

Thus, the second term on the right-hand side of (20) is less than $\epsilon/3$.

Because P^* is a refinement of P, for each $i = 1, \cdots, n$ let P_i^* denote the partition of $[x_{i-1}, x_i]$ induced by P^*. The Refinement Lemma then yields

$$\begin{aligned} 0 \leq {} & U(f, P) - U(f, P^*) = \sum_{i=1}^{n} \big[\overline{m}_i(x_i - x_{i-1}) - U(f, P_i^*)\big]\,, \\ 0 \leq {} & L(f, P^*) - L(f, P) = \sum_{i=1}^{n} \big[L(f, P_i^*) - \underline{m}_i(x_i - x_{i-1})\big]\,. \end{aligned}$$

Because P^ϵ has at most $n^\epsilon - 1$ partition points that are not partition points of P, there are at most $n^\epsilon - 1$ indices i for which $[x_{i-1}, x_i]$ contains at least one partition point of P_i^* within (x_{i-1}, x_i). For all other indices the terms in the above sums are zero. Each of the nonzero terms in the above sums satisfy the bounds

$$\begin{aligned} 0 \leq \quad & \overline{m}_i(x_i - x_{i-1}) - U(f, P_i^*) \leq 2M(x_i - x_{i-1}) < 2M\delta \,, \\ 0 \leq \quad & L(f, P_i^*) - \underline{m}_i(x_i - x_{i-1}) \leq 2M(x_i - x_{i-1}) < 2M\delta \,. \end{aligned}$$

Because there are at most $n^\epsilon - 1$ such terms, we obtain the bounds

$$\begin{aligned} 0 \leq \quad & U(f, P) - U(f, P^*) < n^\epsilon 2M\delta < \frac{\epsilon}{3} \,, \\ 0 \leq \quad & L(f, P^*) - L(f, P) < n^\epsilon 2M\delta < \frac{\epsilon}{3} \,. \end{aligned}$$

This shows the first and third terms on the right-hand side of (20) are less than $\epsilon/3$. □

An immediate consequence of the Partitions Theorem is that there a simple criterion for a sequence of partitions to be Archimedean for any Riemann integrable function.

Theorem 1.4. (Archimedean Sequences) *Every sequence* $\{P^n\}_{n=1}^\infty$ *of partitions of* $[a, b]$ *such that* $|P^n| \to 0$ *as* $n \to \infty$ *is Archimedean for every function* $f : [a, b] \to \mathbb{R}$ *that is Riemann integrable over* $[a, b]$*.*

Proof. Exercise. □

Remark. The condition that $|P^n| \to 0$ as $n \to \infty$ is not necessary for a sequence $\{P^n\}_{n=1}^\infty$ of partitions to be Archimedean. For example, every sequence of partitions is Archimedean for every constant function.

2. Riemann Integrable Functions

In the previous section we defined the Riemann integral and established some of its basic properties. We did not identify a large class of Riemann integrable functions. That is what we will do in this section. Before beginning that task, we remark that there are many functions that are not Riemann integrable.

Exercise. Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

Show that the restriction of f to any closed bounded interval $[a, b]$ is not Riemann integrable.

2.1. **Integrability of Monotonic Functions.**

We first show that the class of Riemann integrable functions includes the class of monotonic functions. Recall that this class is defined as follows.

Definition 2.1. *Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be* nondecreasing *over D provided that*

$$x < y \implies f(x) \leq f(y) \quad \textit{for every } x, y \in D\,.$$

A function $f : D \to \mathbb{R}$ is said to be nonincreasing *over D provided that*

$$x < y \implies f(x) \geq f(y) \quad \textit{for every } x, y \in D\,.$$

A function that is either nondecreasing or nonincreasing is said to be monotonic *over D.*

A function that is monotonic over a closed interval $[a, b]$ is clearly bounded by its endpoint values.

Theorem 2.1. (Monotonic Integrability) *Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then f is Riemann integrable over $[a, b]$. Moreover, for every partition P of $[a, b]$ one has*

$$0 \leq U(f, P) - L(f, P) \leq |P| \, \big| f(b) - f(a) \big| \,. \tag{21}$$

Proof. For any partition $P = [x_0, \cdots, x_n]$ we have the following basic estimate. Because f is monotonic, over each subinterval $[x_{i-1}, x_i]$ one has that

$$\overline{m}_i - \underline{m}_i = \big| f(x_i) - f(x_{i-1}) \big| \,.$$

We thereby obtain

$$\begin{aligned} 0 \le U(f,P) - L(f,P) &= \sum_{i=1}^{n} (\overline{m}_i - \underline{m}_i)\,(x_i - x_{i-1}) \\ &\le |P| \sum_{i=1}^{n} (\overline{m}_i - \underline{m}_i) = |P| \sum_{i=1}^{n} \big| f(x_i) - f(x_{i-1}) \big| \,, \end{aligned}$$

where $|P| = \max\{x_i - x_{i-1} : i = 1, \cdots, n\}$ is the thickness of P. Because f is monotonic, the terms $f(x_i) - f(x_{i-1})$ are either all nonnegative, or all nonpositive. We may therefore pass the absolute value outside the last sum above, which then telescopes. We thereby obtain the estimate

$$\begin{aligned} 0 \le U(f,P) - L(f,P) &\le |P| \sum_{i=1}^{n} \big| f(x_i) - f(x_{i-1}) \big| \\ &= |P| \left| \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \right| \\ &= |P| \, \big| f(b) - f(a) \big| \,. \end{aligned}$$

This establishes (21).

Now let $\epsilon > 0$. Let P be any partition of $[a,b]$ such that $|P|\,|f(b) - f(a)| < \epsilon$. Then by (21) one has

$$0 \le U(f,P) - L(f,P) \le |P| \, \big| f(b) - f(a) \big| < \epsilon \,.$$

Hence, f is Riemann integrable by characterization (2) of the Riemann-Darboux Theorem. □

Remark. It is a classical fact that a monotonic function over $[a,b]$ is continuous at all but at most a countable number of points where it has a jump discontinuity. One example of such a function defined over the interval $[0,1]$ is

$$f(x) = \begin{cases} \dfrac{1}{2^k} & \text{for } \dfrac{1}{2^{k+1}} < x \le \dfrac{1}{2^k} \,, \\ 0 & \text{for} x = 0 \,. \end{cases}$$

One can show that

$$\int_0^1 f = \frac{2}{3} \,.$$

2.2. **Integrability of Continuous Functions.** The class of Riemann integrable functions also includes the class of continuous functions.

Theorem 2.2. (Continuous Integrability) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable over $[a, b]$.*

Remark. Because $[a, b]$ is sequentially compact, the fact that f is continuous over $[a, b]$ implies that it is bounded over $[a, b]$ and that it is uniformly continuous over $[a, b]$. Both of these implications play a role in this theorem. The fact f is bounded is needed to know that the Darboux sums $L(f, P)$ and $U(f, P)$ make sense. The fact f is uniformly continuous will play the central role in our proof.

Proof. Let $\epsilon > 0$. Because f is uniformly continuous over $[a, b]$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a} \quad \text{for every } x, y \in [a, b] .$$

Let $P = [p_0, p_1, \cdots, p_n]$ be any partition of $[a, b]$ such that $|P| < \delta$. Because f is continuous, it takes on extreme values over each subinterval $[p_{i-1}, p_i]$ of P. Hence, for every $i = 1, \cdots, n$ there exist points $\overline{x}_i$ and $\underline{x}_i$ in $[p_{i-1}, p_i]$ such that $\overline{m}_i = f(\overline{x}_i)$ and $\underline{m}_i = f(\underline{x}_i)$. Because $|P| < \delta$ it follows that $|\overline{x}_i - \underline{x}_i| < \delta$, whereby

$$\overline{m}_i - \underline{m}_i = f(\overline{x}_i) - f(\underline{x}_i) < \frac{\epsilon}{b - a} .$$

We thereby obtain

$$\begin{aligned} 0 \leq U(f, P) - L(f, P) &= \sum_{i=1}^{n} (\overline{m}_i - \underline{m}_i)(p_i - p_{i-1}) \\ &\leq \frac{\epsilon}{b - a} \sum_{i=1}^{n} (p_i - p_{i-1}) = \frac{\epsilon}{b - a} (b - a) = \epsilon . \end{aligned}$$

This shows that for every partition P of $[a, b]$ one has

$$|P| < \delta \quad \implies \quad 0 \leq U(f, P) - L(f, P) < \epsilon .$$

But $\epsilon > 0$ was arbitrary. Hence, f is Riemann integrable by the Partition Theorem. □

Exercise. A function $f : [a, b] \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in [a, b]$ one has

$$|f(x) - f(y)| < C\, |x - y|^{\alpha} .$$

Show that for every partition P of $[a, b]$ one has

$$0 \leq U(f, P) - L(f, P) < |P|^{\alpha} C\, (b - a) .$$

2.3. **Linearity and Order for Riemann Integrals.** Linear combinations of Riemann integrable functions are again Riemann integrable. Riemann integrals respect this linearity and order.

2.3.1. *Linearity.* One basic fact about Riemann integrals is that they depend linearly on the integrand. This fact is not completely trivial because we defined the Riemann integral through Darboux sums, which do not depend linearly on the integrand.

Proposition 2.1. (Linearity) *Let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be Riemann integrable over* $[a, b]$. *Let* $\alpha \in \mathbb{R}$. *Then* $f + g$ *and* αf *are Riemann integrable over* $[a, b]$ *with*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \,, \qquad \int_a^b (\alpha f) = \alpha \int_a^b f \,.$$

Proof. A key step towards establishing the additivity is to prove that if P is any partition of $[a, b]$ then

$$L(f,P)+L(g,P) \leq L\big((f+g),P\big) \leq U\big((f+g),P\big) \leq U(f,P)+U(g,P) \,.$$

A key step towards establishing the scalar multiplicity is to prove that if $\alpha > 0$ and P is any partition of $[a, b]$ then

$$L(\alpha f, P) = \alpha L(f, P) \,, \qquad U(\alpha f, P) = \alpha U(f, P) \,.$$

The proof is left as an exercise. □

Remark. It follows immediately from the above proposition that every linear combination of Riemann integrable functions is also Riemann integrable, and that its integral is the same linear combination of the respective integrals. More precisely, if $f_k : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$ for every $k = 1, 2, \cdots, n$ then for every $\{\alpha_k\}_{k=1}^n \subset \mathbb{R}$ one knows that

$$\sum_{k=1}^{n} \alpha_k f_k \quad \text{is Riemann integrable over } [a, b] \,,$$

with

$$\int_a^b \left(\sum_{k=1}^{n} \alpha_k f_k \right) = \sum_{k=1}^{n} \alpha_k \int_a^b f_k \,.$$

2.3.2. *Nonnegativity.* Another basic fact about definite integrals is that they respect nonnegativity of the integrand.

Proposition 2.2. (Nonnegativity) *Let* $f : [a, b] \to \mathbb{R}$ *be nonnegative and Riemann integrable over* $[a, b]$*. Then*

$$0 \leq \int_a^b f \,.$$

Proof. Exercise. ☐

2.3.3. *Order.* The basic comparison property of definite integrals now follows from Propositions 2.1 and 2.2.

Proposition 2.3. (Order) *Let* $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be Riemann integrable over* $[a, b]$*. Let* $f(x) \leq g(x)$ *for every* $x \in [a, b]$*. Then*

$$\int_a^b f \leq \int_a^b g \,.$$

Proof. Exercise. ☐

2.3.4. *Bounds.* The basic bounds on definite integrals now follows from Proposition 2.3.

Proposition 2.4. (Bounds) *Let* $f : [a, b] \to \mathbb{R}$ *be Riemann integrable (and hence, bounded) over* $[a, b]$*. Suppose that* $\mathrm{Range}(f) \subset [\underline{m}, \overline{m}]$*. Then*

$$\underline{m}\,(b - a) \leq \int_a^b f \leq \overline{m}\,(b - a) \,.$$

Moreover,

$$\left| \int_a^b f \right| \leq M\,(b - a) \,,$$

where $M = \sup \big\{ |f(x)| \,:\, x \in [a, b] \big\}$*.*

Proof. Exercise. ☐

2.4. **Nonlinearity.** More general combinations of Riemann integrable functions are again Riemann integrable.

Theorem 2.3. (Compositions) *Let* $f : [a,b] \to \mathbb{R}$ *be Riemann integrable over* $[a,b]$*. Suppose that* $\text{Range}(f) \subset [\underline{m}, \overline{m}]$*. Let* $G : [\underline{m}, \overline{m}] \to \mathbb{R}$ *be continuous. Then* $G(f) : [a,b] \to \mathbb{R}$ *is Riemann integrable over* $[a,b]$*.*

Proof. Because $G : [\underline{m}, \overline{m}] \to \mathbb{R}$ is continuous, it is bounded. Suppose that $\text{Range}(G) \subset [\underline{m}^*, \overline{m}^*]$ where $\underline{m}^* < \overline{m}^*$.

Let $\epsilon > 0$. Because G is uniformly continuous over $[\underline{m}, \overline{m}]$, there exists a $\delta > 0$ such that for every $y, z \in [\underline{m}, \overline{m}]$ one has

$$|y - z| < \delta \implies |G(y) - G(z)| < \frac{\epsilon}{2(b-a)} \,.$$

Because f is Riemann integrable over $[a,b]$ there exists a partition P such that

$$0 \leq U(f,P) - L(f,P) < \frac{\delta\epsilon}{2(\overline{m}^* - \underline{m}^*)} \,.$$

Let $P = [x_0, x_1, \cdots, x_n]$. For every $i = 1, \cdots .n$ define $\underline{m}_i$, $\overline{m}_i$, $\underline{m}_i^*$, and $\overline{m}_i^*$ by

$$\begin{aligned}
\underline{m}_i &= \inf\{f(x) \,:\, x \in [x_{i-1}, x_i]\} \,, \\
\overline{m}_i &= \sup\{f(x) \,:\, x \in [x_{i-1}, x_i]\} \,, \\
\underline{m}_i^* &= \inf\{G(f(x)) \,:\, x \in [x_{i-1}, x_i]\} \,, \\
\overline{m}_i^* &= \sup\{G(f(x)) \,:\, x \in [x_{i-1}, x_i]\} \,.
\end{aligned}$$

The key step is to decompose the indices $i = 1, \cdots .n$ into two sets:

$$I_< = \{i \,:\, \overline{m}_i - \underline{m}_i < \delta\} \,, \qquad I_\geq = \{i \,:\, \overline{m}_i - \underline{m}_i \geq \delta\} \,.$$

We analyze each of these sets separately.

For the "good" set $I_<$ the values of f over $[x_{i-1}, x_i]$ lie in $[\underline{m}_i, \overline{m}_i]$. Because G is continuous, the Extreme-Value Theorem implies that G takes on its inf and sup over $[\underline{m}_i, \overline{m}_i]$, say at the points $\underline{y}_i$ and $\overline{y}_i$ respectively. Because $|\overline{y}_i - \underline{y}_i| < \delta$ for every $i \in I_<$, one has

$$\begin{aligned}
\overline{m}_i^* - \underline{m}_i^* &\leq \sup\{G(y) \,:\, y \in [\underline{m}_i, \overline{m}_i]\} - \inf\{G(y) \,:\, y \in [\underline{m}_i, \overline{m}_i]\} \\
&= G(\overline{y}_i) - G(\underline{y}_i) < \frac{\epsilon}{2(b-a)} \,,
\end{aligned}$$

whereby

$$i \in I_< \implies \overline{m}_i^* - \underline{m}_i^* < \frac{\epsilon}{2(b-a)} \,.$$

The idea is to show that the P is sufficiently refined that the "bad" set $I_{\geq}$ is small. Because $\delta \leq \overline{m}_i - \underline{m}_i$ for every $i \in I_{\geq}$, we have

$$\begin{aligned}\delta \sum_{i \in I_{\geq}} (x_i - x_{i-1}) &\leq \sum_{i \in I_{\geq}} (\overline{m}_i - \underline{m}_i)(x_i - x_{i-1}) \\ &\leq U(f, P) - L(f, P) < \frac{\delta \epsilon}{2(\overline{m}^* - \underline{m}^*)} \,,\end{aligned}$$

whereby

$$\sum_{i \in I_{\geq}} (x_i - x_{i-1}) < \frac{\epsilon}{2(\overline{m}^* - \underline{m}^*)} \,.$$

Upon combining the above estimates we obtain

$$\begin{aligned}0 &\leq U(G(f), P) - L(G(f), P) \\ &= \sum_{i \in I_{<}} (\overline{m}_i^* - \underline{m}_i^*)(x_i - x_{i-1}) + \sum_{i \in I_{\geq}} (\overline{m}_i^* - \underline{m}_i^*)(x_i - x_{i-1}) \\ &< \frac{\epsilon}{2(b-a)} \sum_{i \in I_{<}} (x_i - x_{i-1}) + (\overline{m}^* - \underline{m}^*) \sum_{i \in I_{\geq}} (x_i - x_{i-1}) \\ &< \frac{\epsilon}{2(b-a)} (b-a) + (\overline{m}^* - \underline{m}^*) \frac{\epsilon}{2(\overline{m}^* - \underline{m}^*)} = \epsilon \,.\end{aligned}$$

Because ϵ was arbitrary, $G(f)$ is Riemann integrable by characterization (2) of the Riemann-Darboux Theorem. □

An important consequence of the Composition Theorem is that the product of Riemann integrable functions is also Riemann integrable.

Proposition 2.5. (Product) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. Then product $fg : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$.*

Remark. Taken together the Linearity and Product Propositions show that the class of Riemann integrable functions is an algebra.

Proof. The proof is based on the algebraic identity

$$fg = \tfrac{1}{4}\big((f+g)^2 - (f-g)^2\big) \,.$$

By the Linearity Proposition the functions $f+g$ and $f-g$ are Riemann integrable over $[a, b]$. By Composition Theorem (applied to $G(z) = z^2$) the functions $(f+g)^2$ and $(f-g)^2$ are Riemann integrable over $[a, b]$. Hence, by applying the Linearity Proposition to the above identity, one sees that fg is Riemann integrable over $[a, b]$. □

Remark. We could just as well have built a proof of the Product Lemma based on the identity

$$fg = \tfrac{1}{2}\big((f+g)^2 - f^2 - g^2\big)\,,$$

or the identity

$$fg = \tfrac{1}{2}\big(f^2 + g^2 - (f-g)^2\big)\,.$$

Another consequence of the Composition Theorem is that the absolute-value of a Riemann integrable function is also Riemann integrable. When combined with the Order, Bounds, and Product Propositions 2.3, 2.4, and 2.5, this leads to the following useful bound.

Proposition 2.6. (Absolute-Value) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable over $[a,b]$. Suppose that g is non-negative. Then $|f| : [a,b] \to \mathbb{R}$ is Riemann integrable over $[a,b]$ and satisfies*

$$\left| \int_a^b fg \right| \leq \int_a^b |f|g \leq M \int_a^b g\,,$$

where $M = \sup\{|f(x)| : x \in [a,b]\}$.

Proof. Exercise. □

2.5. **Restrictions and Interval Additivity.** A property of the definite integral that you learned when you first studied integration is interval additivity. In its simplest form this property states that, provided all the integrals exist, for every $a, b, c \in \mathbb{R}$ such that $a < b < c$ one has

$$\int_a^c f = \int_a^b f + \int_b^c f\,. \tag{22}$$

In elementary calculus courses this formula is often stated without much emphasis on implicit integrability assumptions. As we will see below, Riemann integrals have this property. In that setting this formula assumes that f is Riemann integrable over $[a,c]$, and that the restrictions of f to $[a,b]$ and $[b,c]$ are Riemann integrable over those intervals. As the next lemma shows, these last two assumptions follow from the first.

Lemma 2.1. (Restriction) *Let $f : [a,d] \to \mathbb{R}$ be Riemann integrable. Then for every $[b,c] \subset [a,d]$ the restriction of f to $[c,d]$ is Riemann integrable over $[b,c]$.*

Proof. Let $[b, c] \subset [a, d]$. Let $\epsilon > 0$. Because f is Riemann integrable over $[a, d]$ by characterization (2) of the Integrability Theorem there exists a partition P^* of $[a, d]$ such that

$$0 \leq U(f, P^*) - L(f, P^*) < \epsilon\,.$$

By the Refinement Lemma we may assume that b and c are partition points of P^*, otherwise we can simply replace P^* by $P^* \vee [a, b, c, d]$. Let P be the partition of $[b, c]$ induced by P^*. Then

$$0 \leq U(f, P) - L(f, P) \leq U(f, P^*) - L(f, P^*) < \epsilon\,.$$

Hence, f is Riemann integrable over $[b, c]$ by characterization (2) of the Integrability Theorem. □

Now return to the interval additivity formula (22). More interesting from the viewpoint of building up the class of Riemann integrable functions is the fact that if the restrictions of f to $[a, b]$ and $[b, c]$ are Riemann integrable over those intervals then f is Riemann integrable over $[a, c]$. More generally, we have the following.

Proposition 2.7. (Interval Additivity) *Let $P = [p_0, \cdots, p_k]$ be any partition of $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the restriction of f to $[p_{i-1}, p_i]$ is Riemann integrable for every $i = 1, \cdots, k$. Moerover, in that case one has*

$$\int_a^b f = \sum_{i=1}^{k} \int_{p_{i-1}}^{p_i} f\,. \tag{23}$$

Proof. ($\Longrightarrow$) This follows from the Restriction Lemma.

($\Longleftarrow$) Because f is Riemann integrable over $[p_{i-1}, p_i]$ for every $i = 1, \cdots, k$ there exists a partition P_i^* of $[p_{i-1}, p_i]$ such that

$$0 \leq U(f, P_i^*) - L(f, P_i^*) < \frac{\epsilon}{k}\,.$$

Let P^* be the refinement of P such that P_i^* is the induced partition of $[p_{i-1}, p_i]$. One then sees that

$$\begin{aligned} 0 &\leq U(f, P^*) - L(f, P^*) \\ &= \sum_{i=1}^{k} \big(U(f, P_i^*) - L(f, P_i^*) \big) < \sum_{i=1}^{k} \frac{\epsilon}{k} = \epsilon\,. \end{aligned}$$

Hence, by characterization (2) of the Integrability Theorem, f is Riemann integrable over $[a, b]$.

One can use Riemann sums to establish (23). This part of the proof is left as an exercise. □

The restriction $a < b < c$ in the interval additivity formula (22) can be dropped provided one adopts the following convention.

Definition 2.2. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. Then define*

$$\int_b^a f = -\int_a^b f\,.$$

Exercise. Show that (22) holds for every a, b, and $c \in \mathbb{R}$, provided that we adopt Definition 2.2 and f is Riemann integrable over all the intervals involved.

2.6. **Extensions and Piecewise Integrability.** Here we will use interval additivity to build up the claas of Riemann integrable functions. For this approach to be very useful we will need a lemma regarding extensions. To motivate the need for this lemma, let us consider the function f defined over $[-1, 1]$ by

$$f(x) = \begin{cases} x + 2 & \text{for } x \in [-1, 0)\,, \\ 1 & \text{for } x = 0\,, \\ x & \text{for } x \in (0, 1)\,. \end{cases}$$

It is easy to use the Riemann-Darboux Theorem to verify that this function is Riemann integrable with

$$\int_{-1}^1 f = 2\,,$$

yet this fact does not follow directly from other theorems we have proved. For example, f restricted to either $[-1, 0]$ or $[0, 1]$ is neither monotonic nor continuous because of its behavior at $x = 0$. However, our intuition tells us (correctly) that the value of f at 0 should not effect whether or not it is Riemann integrable. The following lemma shows this to be the case if the points in questions are the endpoints of the interval of integration.

Lemma 2.2. (Extension) *Let $f : (a, b) \to \mathbb{R}$ be bounded. Suppose that for every $[c, d] \subset (a, b)$ the restriction of f to $[c, d]$ is Riemann integrable over $[c, d]$. Let $\hat{f} : [a, b] \to \mathbb{R}$ be any extension of f to $[a, b]$. Then $\hat{f}$ is Riemann integrable over $[a, b]$. Moreover, if $\hat{f}_1$ and $\hat{f}_2$ are two such extensions of f then*

$$\int_a^b \hat{f}_1 = \int_a^b \hat{f}_2\,.$$

Proof. Let $\epsilon > 0$. Let $\text{Range}(f) \subset [\underline{m}, \overline{m}]$. Let $\delta > 0$ such that

$$(\overline{m} - \underline{m})\delta < \frac{\epsilon}{3}, \quad \text{and} \quad \delta < \frac{b-a}{2}.$$

Because the restriction of f to $[a+\delta, b-\delta]$ is Riemann integrable, there exists a partition P of $[a+\delta, b-\delta]$ such that

$$0 \leq U(f, P) - L(f, P) < \frac{\epsilon}{3}.$$

Let P^* be the extension of P to $[a, b]$ obtained by adding a and b as partition points. Then

$$\begin{aligned} 0 &\leq U(\hat{f}, P^*) - L(\hat{f}, P^*) \\ &= \big[U(\hat{f}, [a, a+\delta]) - L(\hat{f}, [a, a+\delta])\big] + \big[U(f, P) - L(f, P)\big] \\ &\quad + \big[U(\hat{f}, [a, a+\delta]) - L(\hat{f}, [a, a+\delta])\big] \\ &\leq (\overline{m} - \underline{m})\delta + \frac{\epsilon}{3} + (\overline{m} - \underline{m})\delta < \epsilon. \end{aligned}$$

Hence, the extension $\hat{f}$ is Riemann integrable over $[a, b]$ by characterization (2) of the Riemann-Darboux Theorem.

Now let $\hat{f}_1$ and $\hat{f}_2$ be two extensions of f to $[a, b]$. Let $\{P^n\}_{n=1}^{\infty}$ be any sequence of partitions of $[a, b]$ such that $|P^n| \to 0$ as $n \to \infty$. This sequence is Archimedean for both $\hat{f}_1$ and $\hat{f}_2$ by Theorem 1.4. Let $\{Q^n\}_{n=1}^{\infty}$ be any sequence of associated quadrature points such that neither a nor b are quadrature points. Because $\hat{f}_1(x) = \hat{f}_2(x)$ for every $x \in (a, b)$, we have $R(\hat{f}_1, P^n, Q^n) = R(\hat{f}_2, P^n, Q^n)$ for every $n \in \mathbb{Z}_+$. Therefore the Archimedes-Riemann Theorem yields

$$\int_a^b \hat{f}_1 = \lim_{n\to\infty} R(\hat{f}_1, P^n, Q^n) = \lim_{n\to\infty} R(\hat{f}_2, P^n, Q^n) = \int_a^b \hat{f}_2.$$

□

It is a consequence of the Extension Lemma and interval additivity that two functions that differ at only a finite number of points are the same when is comes to Riemann integrals.

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ such that $g(x) = f(x)$ at all but a finite number of points in $[a, b]$. Then g is Riemann integrable over $[a, b]$ and*

$$\int_a^b g = \int_a^b f.$$

Proof. Exercise. □

Remark. The same cannot be said of two functions that differ at a countable number of points. Indeed, consider the function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

Its restriction to any closed bounded interval $[a, b]$ is not Riemann integrable, yet it differs from $f = 0$ at a countable number of points.

We can now show that all functions that are piecewise monotonic over $[a, b]$ are also Riemann intergrable over $[a, b]$. We first recall the definition of piecewise monotonic function.

Definition 2.3. *A function $f : [a, b] \to \mathbb{R}$ is said to be* piecewise monotonic *if it is bounded and there exists a partition $P = [x_0, \cdots, x_n]$ of $[a, b]$ such that f is monotonic over (x_{i-1}, x_i) for every $i = 1, \cdots, n$.*

Theorem 2.5. *Let $f : [a, b] \to \mathbb{R}$ be piecewise monotonic. Then f is Riemann integrable over $[a, b]$.*

Proof. This follows from Proposition 2.1, the Extension Lemma, and the Interval Additivity Proposition. The details are left as an exercise. □

We can also show that all functions that are piecewise continuous over $[a, b]$ are also intergrable over $[a, b]$. We first recall the definition of piecewise continuous function.

Definition 2.4. *A function $f : [a, b] \to \mathbb{R}$ is said to be* piecewise continuous *if it is bounded and there exists a partition $P = [x_0, \cdots, x_n]$ of $[a, b]$ such that f is continuous over (x_{i-1}, x_i) for every $i = 1, \cdots, n$.*

We remark that piecewise continuous functions are discontinuous at only a finite number of points. Still, the class of piecewise continuous functions includes some fairly wild functions. For example, it contains the function

$$f(x) = \begin{cases} 1 + \sin(1/x) & \text{if } x > 0\,, \\ 4 & \text{if } x = 0\,, \\ -1 + \sin(1/x) & \text{if } x < 0\,, \end{cases}$$

considered over $[-1, 1]$. As wild as this function looks, it is continuous everywhere except at the point $x = 0$.

Theorem 2.6. *Let $f : [a, b] \to \mathbb{R}$ be piecewise continuous. Then f is Riemann integrable over $[a, b]$.*

Proof. This follows from Proposition 2.2, the Extension Lemma, and the Interval Additivity Proposition. The details are left as an exercise. □

2.7. **Lebesgue Theorem.** In this section we state a beautiful theorem of Lebesgue that characterizes those functions that are Riemann integrable. In order to do this we need to introduce the following notion of "very small" subsets of $\mathbb{R}$.

Definition 2.5. *A set $A \subset \mathbb{R}$ is said to have* measure zero *if for every $\epsilon > 0$ there exists a countable collection of open intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that*

$$A \subset \cup_{i=1}^{\infty}(a_i, b_i)\,,$$

and

$$\sum_{i=1}^{\infty}(b_i - a_i) < \epsilon\,.$$

In other words, a set has measure zero if it can be covered by an arbitrarily small countable collection of open intervals, where the size of the countable collection of intervals is defined by the above sum.

Example. Every finite or countable subset of $\mathbb{R}$ has measure zero. In particular, $\mathbb{Q}$ has measure zero. Indeed, consider a countable set $A = \{x_i\}_{i=1}^{\infty} \subset \mathbb{R}$. Let $\epsilon > 0$. Let $r < \frac{1}{3}$ and set $(a_i, b_i) = (x_i - r^i\epsilon, x_i + r^i\epsilon)$ for every $i \in \mathbb{Z}_+$. The collection of open intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ clearly covers A. Moreover,

$$\sum_{i=1}^{\infty}(b_i - a_i) = \sum_{i=1}^{\infty} 2r^i\epsilon = \frac{2r\epsilon}{1-r} < \epsilon\,.$$

The fact that measure zero is a reasonable concept of "very small" is confirmed by the following facts.

Proposition 2.8. *If $B \subset \mathbb{R}$ has measure zero and $A \subset B$ then A has measure zero.*

If $\{A_n\}_{n=1}^{\infty}$ is a collection of subsets of $\mathbb{R}$ each of which has measure zero then

$$A = \cup_{n=1}^{\infty} A_n \quad \textit{has measure zero}\,.$$

Proof. Exercise. □

Of course, one has to show that most sets do not have measure zero.

Exercise. Show that every nonempty open interval (a, b) does not have measure zero.

When the above exercise is combined with the first assertion of Proposition 2.8, it follows that any set that has measure zero does not contain a nonempty open interval.

The following example shows that there are some very interesting sets that have measure zero.

Example. The Cantor set is an uncountable set that has measure zero. The Cantor set is the subset C of the interval $[0,1]$ obtained by sequentially removing "middle thirds" as follows. Define the sequence of sets $\{C_n\}_{n=1}^{\infty}$ as follows

$$\begin{aligned}
C_1 &= [0,1] - (\tfrac{1}{3}, \tfrac{2}{3}) = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]\,, \\
C_2 &= C_1 - (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \\
&= [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]\,, \\
C_3 &= C_2 - (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup (\tfrac{19}{27}, \tfrac{20}{27}) \cup (\tfrac{25}{27}, \tfrac{26}{27}) \\
&= [0, \tfrac{1}{27}] \cup [\tfrac{2}{27}, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{7}{27}] \cup [\tfrac{8}{27}, \tfrac{1}{3}] \\
&\qquad \cup [\tfrac{2}{3}, \tfrac{19}{27}] \cup [\tfrac{20}{27}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{25}{27}] \cup [\tfrac{26}{27}, 1]\,, \\
&\ \vdots
\end{aligned}$$

In general one has

$$C_n = C_{n-1} - \cup_{2k<3^n} (\tfrac{2k-1}{3^n}, \tfrac{2k}{3^n}) \quad \text{for } n > 3\,.$$

One can show by induction that each C_n is the union of 2^n closed intervals each of which have length $1/3^n$. Each C_n is therefore sequentially compact. Moreover, these sets are nested as

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots\,.$$

The Cantor set C is then defined to be the intersection of these sets:

$$C = \cap_{n=1}^{\infty} C_n\,.$$

Being the intersection of nested sequentially compact sets, this set is nonempty. It is harder to show that C is uncountable. We will not do so here. However, from the information given above you should be able to show that C has measure zero.

Exercise. Show the Cantor set has measure zero.

We need one more definition.

Definition 2.6. *Let $A \subset \mathbb{R}$. Let $\mathcal{A}(x)$ be any assertion about a point x. Then we say "$\mathcal{A}$ almost everywhere in A" provided*

$$\big\{x \in A : \mathcal{A}(x) \quad \textit{is false}\big\} \qquad \textit{has measure zero}\,.$$

Roughly speaking, a property holds almost everywhere if it fails on a set of measure zero.

Example. Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

Then $f = 0$ almost everywhere.

We are now ready to state the Lebesgue Theorem.

Theorem 2.7. (Lebesgue) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable over $[a, b]$ if and only if it is continuous almost everywhere in $[a, b]$.*

Proof. The proof is omitted. It is quite involved. One can be found in "Principles of Analysis" by Walter Rudin. □

An immediate consequence of the Lebesgue Theorem is the following.

Corollary 2.1. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then for every nonempty $(c, d) \subset [a, b]$ one has that f is continuous at some point of (c, d).*

Proof. Exercise. □

This result allows us to sharpen our Nonnegativity and Order Propositions regarding Riemann integrals (Propositions 2.2 and 2.3).

Proposition 2.9. (Positivity) *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose that $f \geq 0$ and that $f(x) > 0$ almost everywhere over a nonempty $(c, d) \subset [a, b]$. Then*

$$\int_a^b f > 0\,.$$

Proof. Exercise. □

Proposition 2.10. (Strict Order) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose that $f \leq g$ and that $f(x) < g(x)$ almost everywhere over a nonempty $(c, d) \subset [a, b]$. Then*

$$\int_a^b f < \int_a^b g\,.$$

Proof. Exercise. □

2.8. **Power Rule.** In this section we will derive the so-called power rule for definite integrals — specifically, that for any $p \in \mathbb{R}$ and any $[a, b] \subset \mathbb{R}_+$ one has

$$\int_a^b x^p \, \mathrm{d}x = \begin{cases} \dfrac{b^{p+1} - a^{p+1}}{p+1} & \text{for } p \neq -1\,, \\ \log\left(\dfrac{b}{a}\right) & \text{for } p = -1\,. \end{cases} \tag{24}$$

Of course, you should be familiar with this rule from your previous study of calculus. You should recall that it follows easily from the Fundamental Theorem of Calculus. Here however we will derive it by taking limits of Riemann sums.

We begin with the observation that for any $p \in \mathbb{R}$ the power function $x \mapsto x^p$ is both monotonic and continuous over $\mathbb{R}_+$. It is therefore Riemann integrable over $[a, b]$ either by Theorem 2.1 or by Theorem 2.2. Moreover, a sequence $\{P^n\}_{n=1}^\infty$ of partitions of $[a, b]$ will be Archimedean whenever $|P^n| \to 0$ as $n \to \infty$. The problem therefore reduces to finding such a sequence of partitions and a sequence $\{Q^n\}_{n=1}^\infty$ of associated quadrature sets for which one can show that

$$\lim_{n\to\infty} R(x^p, P^n, Q^n) = \begin{cases} \dfrac{b^{p+1} - a^{p+1}}{p+1} & \text{for } p \neq -1\,, \\ \log\left(\dfrac{b}{a}\right) & \text{for } p = -1\,. \end{cases}$$

We will take two approaches to this problem.

2.8.1. *Uniform Partitions.* Whenever $p \geq 0$ it is clear that the function $x \mapsto x^p$ is Riemann integrable over $[0, b]$. If one uses the uniform partitions over $[0, b]$ given by

$$P^n = [x_0, x_1, \cdots, x_n]\,, \qquad x_i = \frac{ib}{n}\,,$$

and the right-hand rule quadrature sets $Q^n = (x_1, \cdots, x_n)$ then

$$R(x^p, P^n, Q^n) = \frac{b}{n} \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^p = \frac{b^{p+1}}{n^{p+1}} S^p(n)\,,$$

where

$$S^p(n) = \sum_{i=1}^{n} i^p\,.$$

One must therefore show that for every $p \geq 0$ one has

$$\int_0^b x^p \, \mathrm{d}x = \lim_{n\to\infty} \frac{b^{r+1}}{n^{p+1}} S^p(n) = \frac{b^{p+1}}{p+1}\,. \tag{25}$$

Once this is done then for every $[a, b] \subset [0, \infty)$ one has

$$\int_a^b x^p \, \mathrm{d}x = \int_0^b x^p \, \mathrm{d}x - \int_0^a x^p \, \mathrm{d}x = \frac{b^{p+1} - a^{p+1}}{p+1} \,,$$

which agrees with (24) when $a > 0$.

In order to prove (25) one must establish the limit

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \, S^p(n) = \frac{1}{p+1} \,. \tag{26}$$

The details of proving (26) are presented in the book for the cases $p = 0, 1, 2$ with $b = 1$. Few calculus books prove this limit for cases higher than $p = 3$. They usually proceed by first establishing formulas for $S^p(n)$ like

$$\begin{aligned} S^0(n) &= n \,, & S^1(n) &= \frac{n(n+1)}{2} \,, \\ S^2(n) &= \frac{n(n+1)(2n+1)}{6} \,, & S^3(n) &= \frac{n^2(n+1)^2}{4} \,. \end{aligned}$$

The first of these formulas is trivial. The rest are typically verified by an induction argument on n. Given such an explicit formula for $S^p(n)$, establishing (26) is easy. However, this approach does not give any insight into how to obtain these formulas, which grow in complexity as p increases.

Here we will take a different approach that allows us to prove (26) for every $p \in \mathbb{N}$. We will first find a relation that expresses $S^p(n)$ in terms of all the $S^j(n)$ with $j = 0, \cdots, p-1$. Then, instead of using this relation to generate complicated explicit formulas for $S^p(n)$, we will use it to prove (26) via an induction argument on p.

Proof. Clearly $S^0(n) = n$, so that limit (26) holds for $p = 0$. Now assume that for some $q \geq 1$ limit (26) holds for every $p < q$. By a telescoping sum, the binomial formula, and the definition of $S^p(n)$, one obtains the identity

$$\begin{aligned}(n+1)^{q+1}-1 &= \sum_{i=1}^{n}\left[(i+1)^{q+1}-i^{q+1}\right] \\ &= \sum_{i=1}^{n}\sum_{p=0}^{q}\frac{(q+1)!}{p!(q-p+1)!}\,i^p \\ &= \sum_{p=0}^{q}\frac{(q+1)!}{p!(q-p+1)!}\,S^p(n) \\ &= (q+1)\,S^q(n)+\sum_{p=0}^{q-1}\frac{(q+1)!}{p!(q-p+1)!}\,S^p(n)\,.\end{aligned}$$

Upon solving for $S^q(n)$ and dividing by n^{q+1}, we obtain the relation

$$\begin{aligned}\frac{1}{n^{q+1}}\,S^q(n) = \frac{1}{q+1}\Bigg[&\frac{(n+1)^{q+1}}{n^{q+1}}-\frac{1}{n^{q+1}} \\ &-\sum_{p=0}^{q-1}\frac{(q+1)!}{p!(q-p+1)!}\,\frac{1}{n^{q+1}}\,S^p(n)\Bigg]\,.\end{aligned} \tag{27}$$

Because we know

$$\lim_{n\to\infty}\frac{(n+1)^{q+1}}{n^{q+1}}=1\,,\qquad \lim_{n\to\infty}\frac{1}{n^{q+1}}=0\,,$$

and because, by the induction hypothesis, we know

$$\lim_{n\to\infty}\frac{1}{n^{q+1}}\,S^p(n)=0\quad\text{for every } p<q\,,$$

we can pass to the $n\to\infty$ limit in relation (27). We thereby establish that limit (26) holds for $p=q$. □

Remark. The place in our proof that required p to be a natrual number was the point were we used of the binomial formula.

Exercise. Relation (27) can be recast as

$$S^p(n)=\frac{1}{p+1}\left[(n+1)^{p+1}-1-\sum_{j=0}^{p-1}\frac{(p+1)!}{j!(p-j+1)!}\,S^j(n)\right].$$

This can be used to generate explicit formulas for $S^p(n)$ for any $p\geq 1$. To get an idea of how complicated these explict formulas become, start with the fact $S^0(n)=n$ and use the above relation to generate explicit formulas for $S^1(n)$, $S^2(n)$, $S^3(n)$, and $S^4(n)$.

2.8.2. *Nonuniform Partitions.* The difficulty with the previous approach was that the resulting Riemann sums could not generally be evaluated easily. Fermat saw that this difficulty can be elegantly overcome by choosing to use the nonuniform partitions over $[a, b]$ given by

$$P^n = [x_0, x_1, \cdots, x_n] , \qquad x_i = a\left(\frac{b}{a}\right)^{\frac{i}{n}} .$$

By introducing

$$r_n = \left(\frac{b}{a}\right)^{\frac{1}{n}} ,$$

the partition points can be expressed as $x_i = a\, r_n^{\,i}$. If one uses the left-hand rule quadrature sets $Q^n = (x_0, \cdots, x_{n-1})$ then

$$\begin{aligned} R(x^p, P^n, Q^n) &= \sum_{i=0}^{n-1} \left(ar_n^{\,i}\right)^p \left(ar_n^{\,i+1} - ar_n^{\,i}\right) \\ &= a^{p+1}(r_n - 1) \sum_{i=0}^{n-1} r_n^{\,i(p+1)} . \end{aligned}$$

Notice that the last sum is a finite geometric series with ratio $r_n^{(p+1)}$. It can therefore be evaluated as

$$\sum_{i=0}^{n-1} r_n^{\,i(p+1)} = \begin{cases} \frac{r_n^{\,n(p+1)} - 1}{r_n^{(p+1)} - 1} & \text{for } p \neq -1 , \\ n & \text{for } p = -1 . \end{cases}$$

When $p \neq -1$ the Riemann sums are thereby evaluated as

$$\begin{aligned} R(x^p, P^n, Q^n) &= a^{p+1}(r_n - 1)\, \frac{r_n^{\,n(p+1)} - 1}{r_n^{(p+1)} - 1} \\ &= \left(b^{p+1} - a^{p+1}\right) \frac{r_n - 1}{r_n^{(p+1)} - 1} . \end{aligned}$$

Here we have used the fact that $r_n^{\,n} = b/a$ to see that

$$a^{p+1}\left(r_n^{\,n(p+1)} - 1\right) = b^{p+1} - a^{p+1} .$$

Given the above explicit formula for $R(x^p, P^n, Q^n)$, one only needs to show that

$$\lim_{n\to\infty} \frac{r_n - 1}{r_n^{(p+1)} - 1} = \frac{1}{p+1} . \tag{28}$$

Then

$$\lim_{n\to\infty} R(x^p, P^n, Q^n) = \left(b^{p+1} - a^{p+1}\right) \lim_{n\to\infty} \frac{r_n - 1}{r_n^{(p+1)} - 1}$$
$$= \frac{b^{p+1} - a^{p+1}}{p+1} \,,$$

which yields (24) for the case $p \neq -1$. The case $p = -1$ is left as an exercise. □

Exercise. Prove (28).

Exercise. Prove (24) for the case $p = -1$.

Exercise. By taking limits of Riemann sums, show for every positive a and b that

$$\int_0^b a^x \, \mathrm{d}x = \frac{a^b - 1}{\log(a)} \,.$$

Hint: Use uniform partitions.

Remark. Fermat discovered this beautiful derivation of the power rule (24) before Newton and Leibniz developed the fundamental theorems of calculus. In other words, there was no "easy way" to do the problem when Fermat discovered the power rule. It took a genius like Fermat to solve a problem that the "easy way" makes routine. In fact, Fermat's power rule provided an essential clue that led to the development of the "easy way" by Newton and Leibniz.

3. Relating Integration with Differentiation

Both integration and differentiation predate Newton and Leibniz. The definite integral has roots that go back at least as far as Eudoxos and Archimedes, some two thousand years earlier. The derivative goes back at least as far as Fermat. The fact they are connected in some instances was understood by Barrow, who was one of Newton's teachers and whose work was known to Leibniz. The big breakthrough of Newton and Leibniz was the understanding that this connection is general. This realization made the job of computing definite intergrals much easier, which enabled major advances in science, engineering, and mathematics. This connection takes form in what we now call the first and second fundamental theorems of calculus.

3.1. **The First Fundamental Theorem of Calculus.** The business of evaluating integrals by taking limits of Riemann sums is usually either difficult or impossible. However, as you have known since you first studied integration, for many integrands there is a must easier way. We begin with a definition.

Definition 3.1. *Let $f : [a, b] \to \mathbb{R}$. A function $F : [a, b] \to \mathbb{R}$ is said to be a* primitive *or* antiderivative *of f over $[a, b]$ provided*

- *the function F is continuous over $[a, b]$,*
- *there exists a partition $[p_0, \cdots, p_n]$ of $[a, b]$ such that for each $i = 1, \cdots, n$ the function F restricted to (p_{i-1}, p_i) is differentiable and satisfies*

$$F'(x) = f(x) \quad \text{for every } x \in (p_{i-1}, p_i)\,. \tag{29}$$

Remark. Definition 3.1 states that F is continuous and piecewise differentiable over $[a, b]$ and that f is an extension of F' to $[a, b]$.

Remark. Because F may be piecewise differentiable, functions f with a finite number of jump discontinuities may have a primitive. For example, $F(x) = |x|$ is a primitive over $[-1, 1]$ for any of the functions

$$f(x) = \begin{cases} 1 & \text{for } x \in (0, 1]\,, \\ d & \text{for } x = 0\,, \\ -1 & \text{for } x \in [-1, 0)\,, \end{cases}$$

where $d \in \mathbb{R}$ is arbitrary. There is clearly no function F that is differentiable over $[-1, 1]$ such that $F' = f$ because f does not have the intermediate-value property.

Exercise. Let $f : [a, b] \to \mathbb{R}$. Let $F : [a, b] \to \mathbb{R}$ be a primitive of f over $[a, b]$. Let $g : [a, b] \to \mathbb{R}$ such that $g(x) = f(x)$ at all but a finite number of points of $[a, b]$. Show that F is also a primitive of g over $[a, b]$.

It is clear that if F is a primitive of a function f over $[a, b]$ then so is $F + c$ for any constant c. It is a basic fact that a primitive is unique up to this arbitrary additive constant.

Lemma 3.1. *Let $f : [a, b] \to \mathbb{R}$. Let $F_1 : [a, b] \to \mathbb{R}$ and $F_2 : [a, b] \to \mathbb{R}$ be primitives of f over $[a, b]$. Then there exists a constant c such that $F_2(x) = F_1(x) + c$ for every $x \in [a, b]$.*

Proof. Let $G = F_2 - F_1$. We must show that this function is a constant over $[a, b]$. Let P^1 and P^2 be the partitions associated with F_1 and F_2 respectively. Set $P = P^1 \vee P^2$. Express P in terms of its partition points as $P = [p_0, \cdots, p_n]$. For each $i = 1, \cdots, n$ the restriction of G to $[p_{i-1}, p_i]$ is continuous over $[p_{i-1}, p_i]$ and differentiable over (p_{i-1}, p_i) with

$$G'(x) = F_2'(x) - F_1'(x) = f(x) - f(x) = 0 \quad \text{for every } x \in (p_{i-1}, p_i) .$$

It follows from the Mean-Value Theorem that restriction of G to each $[p_{i-1}, p_i]$ is constant c_i over that subinterval. But for each $i = 1, \cdots, n-1$ the point p_i is in the subintervals $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$, whereby $c_i = G(p_i) = c_{i+1}$. Hence, G must be a constant over $[a, b]$. □

Corollary 3.1. *Let $f : [a, b] \to \mathbb{R}$ have a primitive over $[a, b]$. Let $x_o \in [a, b]$ and $y_o \in \mathbb{R}$. Then f has a unique primitive F such that $F(x_o) = y_o$.*

Proof. Exercise. □

Exercise. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1 , \\ -x & \text{for } 1 \le x < 2 , \\ 1 & \text{for } 2 \le x \le 3 . \end{cases}$$

Find F, the primitive of f over $[0, 3]$ specified by $F(0) = 1$.

We are now ready to for the big theorem.

Theorem 3.1. (First Fundamental Theorem of Calculus) *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and have a primitive F over $[a, b]$. Then*

$$\int_a^b f = F(b) - F(a) .$$

Remark. This theorem essentially reduces the problem of evaluating definite integrals to that of finding an explicit primitive of f. While such an explicit primitive cannot always be found, it can be found for a wide class of elementary integrands f.

Proof. We must show that for every partition P of $[a, b]$ one has

$$L(f, P) \leq F(b) - F(a) \leq U(f, P)\,. \tag{30}$$

Let P be an arbitrary partition of $[a, b]$ and let P^* denote the refinement $P \vee [p_0, \cdots, p_n]$. Express P^* in terms of its partition points as $P^* = [x_0, \cdots, x_{n^*}]$. Then for every $i = 1, \cdots, n^*$ one knows that $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continuous, and that $F : (x_{i-1}, x_i) \to \mathbb{R}$ is differentiable. Then by the Lagrange Mean-Value Theorem there exists $q_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(q_i)\,(x_i - x_{i-1}) = f(q_i)\,(x_i - x_{i-1})\,.$$

Because $\underline{m}_i \leq f(q_i) \leq \overline{m}_i$, we see from the above that

$$\underline{m}_i\,(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq \overline{m}_i\,(x_i - x_{i-1})\,.$$

Upon summing these inequalities we obtain

$$L(f, P^*) \leq \sum_{i=1}^{n^*} \big(F(x_i) - F(x_{i-1})\big) \leq U(f, P^*)\,.$$

Because the above sum telescopes, we see that

$$\sum_{i=1}^{n^*} \big(F(x_i) - F(x_{i-1})\big) = F(b) - F(a)\,.$$

The Refinement Lemma therefore yields

$$L(f, P) \leq L(f, P^*) \leq F(b) - F(a) \leq U(f, P^*) \leq U(f, P)\,,$$

from which (30) follows. □

Remark. Notice that the First Fundamental Theorem of Calculus does not require f to be continuous, or even piecewise continuous. Rather it only requires that f be Riemann integrable and have a primitive. Notice too how Definition 3.1 of primitives allows the use of the Lagrange Mean-Value Theorem in the above proof.

The following is an immediate corollary of the First Fundamental Theorem of Calculus.

Corollary 3.2. *Let $F : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$ and differentiable over (a, b). Suppose $F' : (a, b) \to \mathbb{R}$ is bounded over (a, b) and*

Riemann integrable over every $[c,d] \subset (a,b)$. Let f be any extension of F' to $[a,b]$. Then f is Riemann integrable over $[a,b]$ and

$$\int_a^b f = F(b) - F(a)\,.$$

Example. Let F be defined over $[-1,1]$ by

$$F(x) = \begin{cases} x\cos(\log(1/|x|)) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then F is continuous over $[-1,1]$ and differentiable over $[-1,0)\cup(0,1]$ with

$$F'(x) = \cos(\log(1/|x|)) + \sin(\log(1/|x|))\,.$$

As this function is bounded, we have

$$\int_{-1}^{1} \big[\cos(\log(1/|x|)) + \sin(\log(1/|x|))\big]\,\mathrm{d}x = F(1) - F(-1) = 2\,.$$

Here the integrand can be assigned any value at $x = 0$.

3.2. **Second Fundamental Theorem of Calculus.** It is natural to ask if every Riemann integrable function has a primitive. It is clear from the First Fundamental Theorem that if f is Riemann integrable over $[a,b]$ and has a primitive F that one must have

$$F(x) = F(a) + \int_a^x f\,.$$

So given a function f that is Riemann integrable over $[a,b]$, we can define F by the above formula. One then checks if $F'(x) = f(x)$ except at a finite number of points. In general this will not be the case. For example, if $f : [0,1] \to \mathbb{R}$ is the Riemann function given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q} \text{ in lowest terms}\,. \\ 0 & \text{otherwise}\,. \end{cases}$$

This function is continuous are all the irrationals, and so is Riemann integrable by the Lebesgue Theorem. Moreover, one can show that for every $x \in [0,1]$ one has

$$F(x) = \int_0^x f = 0\,.$$

Hence, F is differentiable but $F'(x) \neq f(x)$ at every rational. Therefore F is not a primitive of f. Therfore f has no primitives.

The Second Fundamental Theorem of Calculus shows that the above construction does yield a primitive for a large classes of functions.

Theorem 3.2. (Second Fundamental Theorem of Calculus) *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Define $F : [a, b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f \qquad \text{for every } x \in [a, b]\,.$$

Then $F(a) = 0$, F is continuous over $[a, b]$, and if f is continuous at $c \in [a, b]$ then F is differentiable at c with $F'(c) = f(c)$.

In particular, if f is continuous over $[a, b]$ then F is continuously differentiable over $[a, b]$ with $F' = f$. If f is piecewise continuous over $[a, b]$ then f is an extension of F' to $[a, b]$.

Proof. The fact that $F(a) = 0$ is obvious. We now show that F is (uniformly) continuous over $[a, b]$. Let $\epsilon > 0$. Let $M = \sup\{|f(x)| : x \in [a, b]\}$. Pick $\delta > 0$ such that $M\delta < \epsilon$. Then for every $x, y \in [a, b]$ one has

$$\begin{aligned} &|x - y| < \delta \implies \\ |F(x) - F(y)| = \left| \int_y^x f(t)\,\mathrm{d}t \right| &\leq \left| \int_y^x |f(t)|\,\mathrm{d}t \right| \\ &\leq M \left| \int_y^x \mathrm{d}t \right| = M\,|x - y| < M\,\delta < \epsilon\,. \end{aligned}$$

This shows that F is uniformly continuous over $[a, b]$.

Now let f be continuous at $c \in [a, b]$. Let $\epsilon > 0$. Because f is continuous at c there exists a $\delta > 0$ such that for every $z \in [a, b]$ one has

$$|z - c| < \delta \implies \big|f(z) - f(c)\big| < \epsilon\,.$$

Because $f(c)$ is a constant, for every $x \in [a, b]$ such that $x \neq c$ one has

$$f(c) = \frac{1}{x - c} \int_c^x f(c)\,\mathrm{d}z\,.$$

It follows that

$$\begin{aligned} \frac{F(x) - F(c)}{x - c} - f(c) &= \frac{1}{x - c} \int_c^x f(z)\,\mathrm{d}z - \frac{1}{x - c} \int_c^x f(c)\,\mathrm{d}z \\ &= \frac{1}{x - c} \int_c^x \big(f(z) - f(c)\big)\,\mathrm{d}z\,. \end{aligned}$$

Therefore for every $x \in [a,b]$ one has

$$\begin{aligned} &0 < |x - c| < \delta \implies \\ &\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \left| \frac{1}{x - c} \int_c^x \big(f(z) - f(c) \big) \, \mathrm{d}z \right| \\ &\qquad\qquad \leq \frac{1}{|x - c|} \left| \int_c^x \big| f(z) - f(c) \big| \, \mathrm{d}z \right| \\ &\qquad\qquad < \frac{\epsilon}{|x - c|} \left| \int_c^x \mathrm{d}z \right| = \frac{\epsilon}{|x - c|} \, |x - c| = \epsilon \,. \end{aligned}$$

But this is the ϵ-δ characterization of

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c} = f(c) \,.$$

Hence, F is differentiable at c with $F'(c) = f(c)$.

The remainder of the proof is left as an exercise. □

Remark. Roughly speaking, the First and Second Fundamental Theorems of Calculus respectively state that

$$F(x) = F(a) + \int_a^x F'(t) \, \mathrm{d}t \,, \qquad f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \, \mathrm{d}t \,.$$

In words, the first states that integration undoes differentiation (up to a constant), while the second states that differentiation undoes integration. In other words, integration and differentiation are (nearly) inverses of each other. This is the realization that Newton and Leibniz had.

Remark. Newton and Leibniz were influenced by Barrow. He had proved the Second Fundamental Theorem for the special case where f was continuous and monotonic. This generalized Fermat's observation that the Second Fundamental Theorem holds for the power functions x^p, which are continuous and monotonic over $x > 0$. Of course, neither Barrow's statement nor his proof of this theorem were given in the notation we use today. Rather, they were given in a highly geometric setting that was commonly used at the time. This made it harder to see that his result could be generalized further. You can get an idea of what he did by assuming that f is nondecreasing and continuous over $[a,b]$ and drawing the picture that goes with the inequality

$$a \leq x < y \leq b \implies f(x) \leq \frac{F(y) - F(x)}{y - x} \leq f(y) \,,$$

where $F(x) = \int_a^x f$. By letting $y \to x$ while using the continuity of f, one obtains $F'(x) = f(x)$.

3.3. **Integration by Parts.** An important consequence of the First Fundamental Theorem of Calculus and the Product Rule for derivatives is the following lemma regarding integration by parts.

Lemma 3.2. (Integration by Parts) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable and have primitives F and G respectively over $[a,b]$. Then Fg and Gf are Riemann integrable over $[a,b]$ and*

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b Gf \,. \tag{31}$$

Proof. The functions F and G are Riemann integrable over $[a,b]$ because they are continuous. The functions Fg and Gf are therefore Riemann integrable over $[a,b]$ by the Product Lemma.

The function FG is continuous over $[a,b]$. Let P and Q be the partitions of $[a,b]$ associated with F and G respectively. Let $R = P \vee Q$. Express R in terms of its partition points as $P = [r_0, \cdots, r_n]$. Then for every $i = 1, \cdots, n$ the function FG is differentiable over (r_{i-1}, r_i) with (by the Product Rule)

$$\begin{aligned}
(FG)'(x) &= F(x)G'(x) + G(x)F'(x) \\
&= F(x)g(x) + G(x)f(x) \\
&= (Fg + Gf)(x) \qquad\qquad \text{for every } x \in (r_{i-1}, r_i)\,.
\end{aligned}$$

Therefore FG is a primitive of $Fg + Gf$ over $[a,b]$. Equation (31) then follows by the First Fundamental Theorem of Calculus and the Additivity Lemma. □

In the case where f and g are continuous over $[a,b]$ then the Second Fundamental Theorem of Calculus implies that f and g have primitives F and G that are piecewise continuously differentiable over $[a,b]$. In that case integration by parts reduces to the following.

Corollary 3.3. *Let $F : [a,b] \to \mathbb{R}$ and $G : [a,b] \to \mathbb{R}$ be continuously differentiable over $[a,b]$. Then*

$$\int_a^b FG' = F(b)G(b) - F(a)G(a) - \int_a^b GF' \,.$$

3.4. **Substitution.** An important consequence of the First Fundamental Theorem of Calculus and the Chain Rule for derivatives is the following lemma regarding changing the variable of integration in a definite integral by monotonic substitution $y = G(x)$.

Proposition 3.1. (Monotonic Substitution) *Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and have a primitive G that is increasing over $[a, b]$. Let $f : [G(a), G(b)] \to \mathbb{R}$ be Riemann integrable and have a primitive F over $[G(a), G(b)]$ such that $f(G)g$ is Riemann integrable over $[a, b]$. Then one has the change of variable formula*

$$\int_{G(a)}^{G(b)} f = \int_a^b f(G)g\,. \tag{32}$$

Remark. If we show the variables of integration explicitly then the change of variable formula (32) takes the form

$$\int_{G(a)}^{G(b)} f(y)\,\mathrm{d}y = \int_a^b f(G(x))g(x)\,\mathrm{d}x\,.$$

Remark. The assumption that G is increasing over $[a, b]$ could have equivalently been stated as g is positive almost everywhere over $[a, b]$. Because G is a primitive, it is continuous as well as increasing. Its range is therefore the interval $[G(a), G(b)]$, the interval overwhich f and F are assumed to be defined. This insures the compositions $f(G)$ and $F(G)$ are defined over $[a, b]$.

Proof. Let $P = [p_0, \cdots, p_l]$ be the partition of $[G(a), G(b)]$ associated with the primitive F. Let $Q = [q_0, \cdots, q_m]$ be the partition of $[a, b]$ associated with the primitive G. Because $G : [a, b] \to [G(a), G(b)]$ is increasing, $G^{-1}(P) = [G^{-1}(p_0), \cdots, G^{-1}(p_l)]$ is a partition of $[a, b]$. Consider the partition $R = Q \vee G^{-1}(P)$ of $[a, b]$. Express R in terms of its partition points as $R = [r_0, \cdots, r_n]$.

The function $F(G) : [a, b] \to \mathbb{R}$ is continuous over $[a, b]$. Then for every $i = 1, \cdots, n$ the function $F(G)$ is differentiable over (r_{i-1}, r_i) with (by the Chain Rule)

$$F(G)'(x) = F'(G(x))\,G'(x) = f(G(x))\,g(x) \quad \text{for every } x \in (r_{i-1}, r_i)\,.$$

Therefore $F(G)G$ is a primitive of $f(G)g$ over $[a, b]$. Because $f(G)g$ is Riemann integrable over $[a, b]$, the First Fundamental Theorem of Calculus implies

$$\int_a^b f(G)g = F(G)(b) - F(G)(a) = F(G(b)) - F(G(a))\,.$$

On the other hand, because f is Riemann integrable and F is a primitive of f over $[G(a), G(b)]$, the First Fundamental Theorem of Calculus also implies

$$\int_{G(a)}^{G(b)} f = F(G(b)) - F(G(a))\,.$$

The change of variable formula (32) immediately follows from the last two equations. □

Remark. The assumption that G is increasing over $[a, b]$ could have been replaced by the assumption that G is decreasing over $[a, b]$. In that case the interval $[G(b), G(a)]$ replaces the interval $[G(a), G(b)]$ in the hypotheses regarding f and F, but the change of variable formula (32) remains unchanged.

Exercise. The assumption that G is increasing over $[a, b]$ in Proposition 3.1 can be weakened to the assumption that G is nondecreasing over $[a, b]$. Prove this slightly strengthend lemma. The proof can be very similar to the one given above, however you will have to work a bit harder to show that $F(G)$ is a primitive of $f(G)g$ over $[a, b]$.

It is natural to ask whether one needs a hypothesis like G is monotonic over $[a, b]$ in order to establish the change of variable formula (32). Indeed, one does not. However, without it one must take care to insure the compositions $f(G)$ and $F(G)$ are defined over $[a, b]$, to insure that $F(G)$ is a primitive of $f(G)g$ over $[a, b]$, and to insure that $f(G)g$ is Riemann integrable over $[a, b]$. Here is a simple way to do that.

Proposition 3.2. (Nonmonotonic Substitution) *Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and have a primitive G over $[a, b]$. Suppose that* $\text{Range}(G) \subset [\underline{m}, \overline{m}]$ *and let $f : [\underline{m}, \overline{m}] \to \mathbb{R}$ be continuous over $[\underline{m}, \overline{m}]$. Then the change of variable formula (32) holds.*

Proof. By the Second Fundamental Theorem of Calculus f has a continuously differentiable primitive F over $[\underline{m}, \overline{m}]$. It is then easy to show that $F(G)G$ is a primitive of $f(G)g$ over $[a, b]$. Because $f(G)$ is continuous (hence, Riemann integrable) while g is Riemann integrable over $[a, b]$, it follows from the Product Lemma that $f(G)g$ is Riemann integrable over $[a, b]$. The rest of the proof proceeds as that of Proposition 3.1, except here the partitions P, Q, and R are trivial. □

3.5. **Integral Mean-Value Theorem.** We will now give a useful theorem that a first glance does not seem to have a connection with either Fundamental Theorem of Calculus or with the Mean-Value Theorem for differentiable functions. However, as will be explained later, there is a connection.

Theorem 3.3. (Integral Mean-Value) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and positive almost* *everywhere over* $[a,b]$. *Then there exists a point* $p \in (a,b)$ *such that*

$$\int_a^b fg = f(p) \int_a^b g \tag{33}$$

Proof. Because f is continuous over $[a,b]$, the Extreme-Value Theorem there exists points $\underline{x}$ and $\overline{x} \in [a,b]$ such that

$$f(\underline{x}) = \inf\big\{f(x) \,:\, x \in [a,b]\big\}, \qquad f(\overline{x}) = \sup\big\{f(x) \,:\, x \in [a,b]\big\}.$$

Then

$$f(\underline{x}) \leq f(x) \leq f(\overline{x}) \quad \text{for every } x \in [a,b],$$

which, because g is nonnegative, implies that

$$f(\underline{x}) \int_a^b g \leq \int_a^b fg \leq f(\overline{x}) \int_a^b g.$$

If $f(\underline{x}) = f(\overline{x})$ then f is constant and (33) holds for every $c \in (a,b)$. So suppose $f(\underline{x}) < f(\overline{x})$.

Because $f(\underline{x}) < f(\overline{x})$ and because f is continuous there exists $[c,d] \subset [a,b]$ such that $\underline{x} \in [c,d]$ and that $f(x) < \frac{1}{2}(f(\underline{x}) + f(\overline{x}))$. Then

$$f(\overline{x}) - f(x) > \tfrac{1}{2}\big(f(\overline{x}) - f(\underline{x})\big) > 0 \quad \text{for every } x \in (c,d).$$

Because $(f(\overline{x}) - f)g \geq 0$, and because $(f(\overline{x}) - f(x))g(x) > 0$ almost everywhere over the nonempty interval (c,d), the Positivity Lemma implies

$$0 < \int_a^b \big(f(\overline{x}) - f\big)g = f(\overline{x}) \int_a^b g - \int_a^b fg.$$

In a similar manner we can argue that

$$0 < \int_a^b fg - f(\overline{x}) \int_a^b g.$$

Because g is positive almost everywhere over $[a,b]$, the Positivity Lemma also implies that $\int_a^b g > 0$. Therefore, we see that

$$f(\underline{x}) < \frac{\int_a^b fg}{\int_a^b g} < f(\overline{x}).$$

Because f is continuous, the Intermediate-Value Theorem implies there exists a p between $\underline{x}$ and $\overline{x}$ such that (33) holds. $\square$

Remark. The connection of this theorem to both the First and Second Fundamental Theorem of Calculus and to the Lagrange Mean-Value Theorem for differentiable functions is seen in the case $g = 1$. Then by the Second Fundamental Theorem of Calculus f has a continuously differentiable primitive F. The Lagrange Mean-Value Theorem applied to F then yields a $p \in (a,b)$ such that

$$F(b) - F(a) = F'(p)(b-a)\,.$$

By the First Fundamental Theorem of Calculus we therefore have

$$\int_a^b f = F(b) - F(a) = F'(p)(b-a) = f(p)(b-a)\,.$$

This is just (33) for the case $g = 1$. In other words, this case of the Integral Mean-Value Theorem is just the Mean-Value Theorem for differentiable functions applied to a primitive.

3.6. **Cauchy Riemainder Theorem.** Recall that if f is n-times differentiable over an interval (a,b) and $c \in (a,b)$ then the n^{th} Taylor polynomial approximation of f at c is given by

$$T_c^n f(x) = \sum_{k=0}^{n} f^{(k)}(c)\frac{(x-c)^k}{k!}\,. \tag{34}$$

Recall too that if f is $n+1$-times continuously differentiable over the interval (a,b) then the Lagrange Remainder Theorem states that for every $x \in (a,b)$ there exists a point p between c and x such that

$$f(x) = T_c^n f(x) + f^{(n+1)}(p)\frac{(x-c)^{n+1}}{(n+1)!}\,. \tag{35}$$

Our proof of the Lagrange Remainder Theorem was based on a direct application of the Mean-Value Theorem.

Here we give an alternative representation of the remainder due to Cauchy. Its proof is based on a direct application of the First Fundemental Theorem of Calculus, the proof of which also rests on the Mean-Value Theorem. We will see that the resulting representation contains more information than that of Lagrange.

Theorem 3.4. (Cauchy Remainder) *Let f be $n+1$-times differentiable over the interval (a,b) and let $f^{(n+1)}$ be Riemann integrable over every closed subinterval of (a,b). Let $c \in (a,b)$. Then for every $x \in (a,b)$ one has*

$$f(x) = T_c^n f(x) + \int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t\,. \tag{36}$$

Proof. Let $x \in (a,b)$. Then define $F : (a,b) \to \mathbb{R}$ by

$$F(t) = T_t^n f(x) = f(t) + \sum_{k=1}^{n} f^{(k)}(t)\frac{(x-t)^k}{k!} \quad \text{for every } t \in (a,b)\,. \tag{37}$$

Clearly F is differentiable over (a,b) with (notice the telescoping sum)

$$\begin{aligned} F'(t) &= f'(t) + \sum_{k=1}^{n} \left[f^{(k+1)}(t)\frac{(x-t)^k}{k!} - f^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!} \right] \\ &= f^{(n+1)}(t)\frac{(x-t)^n}{n!} \,. \end{aligned}$$

Because c and x are in (a,b) and because $f^{(n+1)}$ (and hence F') is Riemann integrable over every closed subinterval of (a,b), the First Fundamental Theoren of Calculus yields

$$F(x) - F(c) = \int_c^x F'(t)\,\mathrm{d}t = \int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t \,. \tag{38}$$

However, it is clear from definition (37) of $F(t)$ that

$$F(x) = f(x)\,, \quad \text{while} \quad F(c) = T_c^n f(x)\,.$$

Formula (36) therefore follows from (38). □

The Lagrange Remainder Theorem can be derived from Cauchy's. If one assumes that $f^{(n+1)}$ is continuous over (a,b) then by the Integral Mean-Value Theorem for each $x \in (a,b)$ there exists a point p between c and x such that

$$\int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t = f^{(n+1)}(p)\int_c^x \frac{(x-t)^n}{n!}\,\mathrm{d}t \,.$$

A direct calculation then shows that

$$\int_c^x \frac{(x-t)^n}{n!}\,\mathrm{d}t = \frac{(x-c)^{n+1}}{(n+1)!}\,,$$

whereby

$$\int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t = f^{(n+1)}(p)\frac{(x-c)^{n+1}}{(n+1)!}\,. \tag{39}$$

When this is placed into Cauchy's formula (36) one obtains Lagrange's formula (35).

Remark. One cannot derive the Cauchy Remainder Theorem from that of Lagrange. This is because the Lagrange theorem only tells you that the point p appearing in (35) lies between c and x while the Cauchy theorem provides you with the explicit formula (36) for the remainder.

Remark. The only way to bound the Taylor remainder using the Lagrange form (35) is to use uniform bounds on $f^{(n+1)}(p)$ over all p that lie between c and x. While this approach is sufficient for some tasks (like showing that the formal Taylor expansions of e^x, $\cos(x)$, and

$\sin(x)$ converge to those functions for every $x \in \mathbb{R}$), it fails for other tasks. However, if you are able to obtain suitable pointwise bounds on $f^{(n+1)}(t)$ for every t between c and x then the Cauchy form (36) can sometimes yield bounds on the Taylor remainder that are sufficient for those tasks. This remark is illustrated by the following example.

Example. Let $f(x) = \log(1+x)$ for every $x > -1$. Then

$$f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k} \quad \text{for every } x > -1 \text{ and } k \in \mathbb{Z}_+ .$$

The formal Taylor expansion of f about 0 is therefore

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k .$$

The Absolute Ratio Test shows that this series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = -1$ the series is the negative of the harmonic series, and therefore diverges. For $x = 1$ the Alternating Series Test shows the series converges. We therefore conclude that the series converges if and only if $x \in (-1, 1]$. These arguments do not show however that the sum of the series is $f(x)$. This requires showing that for every $x \in (-1, 1]$ the Taylor remainder $f(x) - T_0^n f(x)$ vanishes as $n \to \infty$.

First let us approach this problem using the Lagrange form of the remainder (35): there exists a p between 0 and x such that

$$f(x) - T_0^n f(x) = f^{(n+1)}(p)\frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n}{n+1}\left(\frac{x}{1+p}\right)^{n+1} .$$

If $x \in (0, 1]$ then $p \in (0, x)$ and we obtain the bound

$$\big|f(x) - T_0^n f(x)\big| < \frac{1}{n+1} x^{n+1} .$$

This bound clearly vanishes as $n \to \infty$ for every $x \in (0, 1]$. On the other hand, if $x \in (-1, 0)$ then $p \in (-|x|, 0)$ and we obtain the bound

$$\big|f(x) - T_0^n f(x)\big| < \frac{1}{n+1}\left(\frac{|x|}{1-|x|}\right)^{n+1} .$$

This bound will only vanish as $n \to \infty$ for $x \in [-\frac{1}{2}, 0)$. This approach leaves open the question for $x \in (-1, -\frac{1}{2})$.

Now let us approach this problem using the Cauchy form of the remainder (36):

$$f(x) - T_0^n f(x) = \int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}\,\mathrm{d}t = (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t$$

Let us only consider the case $x \in (-1, 0)$. Consider the substitution

$$s = \frac{t - x}{1 + t} = 1 - \frac{1 + x}{1 + t}\,, \qquad t + 1 = \frac{1 + x}{1 - s}\,, \qquad \mathrm{d}t = \frac{1 + x}{(1 - s)^2}\,\mathrm{d}s\,.$$

Notice that s decreases from $-x$ $(= |x|)$ to 0 as t decreases from 0 to x. Hence, because

$$\frac{1}{1 - s} < \frac{1}{1 + x} \quad \text{for every } s \in (0, |x|)\,,$$

we obtain the bound

$$\big|f(x) - T_0^n f(x)\big| = \int_0^{|x|} \frac{s^n}{1 - s}\,\mathrm{d}s < \frac{1}{1 + x} \int_0^{|x|} s^n\,\mathrm{d}s = \frac{1}{1 + x}\,\frac{|x|^{n+1}}{n + 1}\,.$$

This bound clearly vanishes as $n \to \infty$ for every $x \in (-1, 0)$.

Collecting all of our results, we have shown that

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{for every } x \in (-1, 1]\,,$$

and that the series diverges for all other values of x.

Exercise. Let $p \leq -1$. Let $f(x) = (1 + x)^p$ for every $x > -1$. Then

$$f^{(k)}(x) = p(p-1)\cdots(p-k+1)(1+x)^{p-k} \quad \text{for every } x > -1 \text{ and } k \in \mathbb{Z}_+\,.$$

The formal Taylor expansion of f about 0 is therefore

$$1 + \sum_{k=1}^{\infty} \frac{p(p - 1)\cdots(p - k + 1)}{k!} x^k\,.$$

Show that this series converges to $f(x)$ for every $x \in (-1, 1)$ and diverges for all other values of x.

Exercise. Let $f(x) = (1 + x)^{-\frac{1}{2}}$ for every $x > -1$. Then

$$f^{(k)}(x) = (-1)^k \frac{(2k)!}{2^{2k} k!} (1 + x)^{-\frac{2k+1}{2}} \quad \text{for every } x > -1 \text{ and } k \in \mathbb{Z}_+\,.$$

The formal Taylor expansion of f about 0 is therefore

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k}(k!)^2} x^k\,.$$

Show that this series converges to $f(x)$ for every $x \in (-1, 1]$ and diverges for all other values of x. You can use the fact that

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}}{n!} \left(\frac{n}{e}\right)^n = 1\,.$$