Math 246, Fall 2008, Professor David Levermore

1. Exact Differential Forms and Integrating Factors

Let us ask the following question. Given a first-order ordinary equation in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)\,, \tag{1}$$

when do its solutions satisfy a relation of the form

$$H(x,y) = C \qquad \text{for some constant } C\,?$$

Such an $H(x,y)$ is called an *integral* of (1).

This question is easily answered if we assume that all functions involved are as differntiable as we need. Suppose that such an $H(x,y)$ exists, and that $y = Y(x)$ is a solution of differential equation (1). Then

$$H(x, Y(x)) = H(x_o, Y(x_o))\,,$$

where x_o is any point in the interval of existence of Y. By differentiating this equation with respect to x we find that

$$\partial_x H(x, Y(x)) + Y'(x)\partial_y H(x, Y(x)) = 0\,.$$

Therefore, wherever $\partial_y H(x, Y(x)) \neq 0$ we see that

$$Y'(x) = -\frac{\partial_x H(x, Y(x))}{\partial_y H(x, Y(x))}\,.$$

For this to hold for every solution of (1), we must have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial_x H(x,y)}{\partial_y H(x,y)}\,,$$

or equivalently

$$f(x,y) = -\frac{\partial_x H(x,y)}{\partial_y H(x,y)}\,, \tag{2}$$

wherever $\partial_y H(x,y) \neq 0$. The question then arises as to whether we can find an $H(x,y)$ such that (2) holds for any given $f(x,y)$? It turns out that this cannot always be done. In this section we explore how to seek such an $H(x,y)$.

1.1. **Exact Differential Forms.**

The starting point is to write equation (1) in a so-called *differential form*

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,, \tag{3}$$

where

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}\,.$$

There is not a unique way to do this. Just pick one that looks natural. If you are lucky then there will exist a function $H(x,y)$ such that

$$\partial_x H(x,y) = M(x,y)\,, \qquad \partial_y H(x,y) = N(x,y)\,. \tag{4}$$

It turns out that there is an easy test you can apply to find out if you are lucky! It derives from the fact that "mixed partials commute" — namely, the fact that for any twice differentiable $H(x,y)$ one has

$$\partial_y\big(\partial_x H(x,y)\big) = \partial_x\big(\partial_y H(x,y)\big)\,.$$

This fact implies that if (4) holds for some twice differentiable $H(x,y)$ then $M(x,y)$ and $N(x,y)$ satisfy

$$\partial_y M(x,y) = \partial_y\big(\partial_x H(x,y)\big) = \partial_x\big(\partial_y H(x,y)\big) = \partial_x N(x,y)\,.$$

This motivates the following definition.

Definition: The differential form (3) is said to be *exact* whenever $M(x,y)$ and $N(x,y)$ satisfy

$$\partial_y M(x,y) = \partial_x N(x,y)\,.$$

We have showed that if there exists an $H(x,y)$ such that (4) holds then the differential form (3) is exact, The remarkable fact is that the converse holds too. Namely, if the differential form (3) is exact then there exists an $H(x,y)$ such that (4) holds. Moreover, the problem of finding $H(x,y)$ is reduced to evaluating two integrals. We illustrate this fact with examples.

Example: Solve the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \quad y(0) = 0\,.$$

Solution: Express this equation in the differential form

$$(e^x y + 2x)\,\mathrm{d}x + (2y + e^x)\,\mathrm{d}y = 0\,.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x \qquad = \qquad \partial_x(2y + e^x) = e^x\,.$$

We can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = e^x y + 2x\,, \qquad \partial_y H(x,y) = 2y + e^x\,. \tag{5}$$

You can now integrate either equation, and plug the result into the other equation to obtain a second equation to integrate.

If we first integrate the first equation in (5) then we find that

$$H(x,y) = \int (e^x y + 2x)\,\mathrm{d}x = e^x y + x^2 + h(y)\,.$$

Here we are integrating with respect to x while treating y as a constant. The function $h(y)$ is the "constant of integration". We plug this expression for $H(x,y)$ into the second equation in (5) to obtain

$$e^x + h'(y) = \partial_y H(x,y) = 2y + e^x\,.$$

This reduces to $h'(y) = 2y$. Notice that this equation for $h'(y)$ only depends on y. Taking $h(y) = y^2$, we see the general solution satisfies

$$H(x,y) = e^x y + x^2 + y^2 = C\,.$$

The initial condition $y(0) = 0$ implies that

$$C = e^0 \cdot 0 + 0^2 + 0^2 = 0\,.$$

Therefore

$$y^2 + e^x y + x^2 = 0\,.$$

The quadratic formula then yields

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2},$$

where the positive square root is taken so that solution satisfies the initial condition. This is a solution wherever $e^{2x} > 4x^2$. □

Alternative Solution: If we first integrate the second equation in (5) then we find that

$$H(x,y) = \int (2y + e^x)\,\mathrm{d}y = y^2 + e^x y + h(x)\,.$$

Here we are integrating with respect to y while treating x as a constant. The function $h(x)$ is the "constant of integration". We plug this expression for $H(x,y)$ into the first equation in (5) to obtain

$$e^x y + h'(x) = \partial_x H(x,y) = e^x y + 2x\,.$$

This reduces to $h'(x) = 2x$. Notice that this equation for $h'(x)$ only depends on x. Taking $h(x) = x^2$, we see the general solution satisfies

$$H(x,y) = e^x y + x^2 + y^2 = C\,.$$

Because this is the same relation for the general solution that we had found previously, the evaluation of C is done as before. □

The points to be made here are the following:

- It does not matter which equation in (4) that you integrate first.
- If you integrate with respect to x first then the "constant of integration" $h(y)$ will depend on y and the equation for $h'(y)$ should only depend on y.
- If you integrate with respect to y first then the "constant of integration" $h(x)$ will depend on x and the equation for $h'(x)$ should only depend on x.
- In either case, if your equation for h' involves both x and y you have made a mistake!

Sometimes the differential equation will be given to you in already in differential form. In that case, use that form as the starting point.

Example: Give an implicit general solution to the differential equation

$$(xy^2 + y + e^x)\,\mathrm{d}x + (x^2 y + x)\,\mathrm{d}y = 0\,.$$

Solution: This differential form is exact because

$$\partial_y(xy^2 + y + e^x) = 2xy + 1 \qquad = \qquad \partial_x(x^2 y + x) = 2xy + 1\,.$$

You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = xy^2 + y + e^x\,, \qquad \partial_y H(x,y) = x^2 y + x\,.$$

By integrating the second equation you obtain

$$H(x,y) = \int (x^2 y + x)\,\mathrm{d}y = \tfrac{1}{2}x^2 y^2 + xy + h(x)\,.$$

When you plug this expresion for $H(x,y)$ into the first equation you obtain

$$xy^2 + y + h'(x) = \partial_x H(x,y) = xy^2 + y + e^x\,,$$

which yields $h'(x) = e^x$. (Notice that this only depends on x!) Taking $h(x) = e^x$, the general solution is

$$\tfrac{1}{2}x^2 y^2 + xy + e^x = C\,.$$

□

Remark: The book gives formulas for $H(x,y)$ in terms of definite integrals. These formulas just encode the two steps given above. I strongly recommend that you simply learn the steps rather than memorize the formulas in the book.

Remark: Our recipe for separable equations can be viewed as a special case of our recipe for exact differential forms. Consider the separable first-order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)\,.$$

It has the differential form

$$f(x)\,\mathrm{d}x - \frac{1}{g(y)}\,\mathrm{d}y = 0\,.$$

This form is exact because

$$\partial_y f(x) = 0 \qquad = \qquad \partial_x \frac{1}{g(y)}\,.$$

You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = f(x)\,, \qquad \partial_y H(x,y) = \frac{1}{g(y)}\,.$$

Indeed, you find that

$$H(x,y) = F(x) - G(y)\,, \qquad \text{where } F'(x) = f(x) \text{ and } G'(y) = \frac{1}{g(y)}\,.$$

The general solution thereby satisfies

$$H(x,y) = F(x) - G(y) = C\,.$$

This is precisely the recipe we derived earlier.

1.2. **Integrating Factors.** Suppose you had considered the differential form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0\,,$$

and found that is not exact. Just because you were unlucky the first time, do not give up! Rather, seek a nonzero function $\mu(x,y)$ such that the differential form

$$\mu(x,y)M(x,y)\,\mathrm{d}x + \mu(x,y)N(x,y)\,\mathrm{d}y = 0 \quad \text{is exact!} \tag{6}$$

This means that $\mu(x,y)$ must satisfy

$$\partial_y\big(\mu(x,y)M(x,y)\big) = \partial_x\big(\mu(x,y)N(x,y)\big)\,.$$

This means that μ satisfies

$$M(x,y)\partial_y\mu + [\partial_y M(x,y)]\mu = N(x,y)\partial_x\mu + [\partial_x N(x,y)]\mu\,. \tag{7}$$

This is a first-order linear partial differential equation for μ. Finding its general solution is equivalent to finding the general solution of the original ordinary differential equation. Fortunately, you do not need this general solution. All you need is one nonzero solution. Such a μ is called an *integrating factor* for the differential form (6).

A trick that sometimes yields a solution of (7) is to assume either that μ is only a function of x, or that μ is only a function of y. When μ is only a function of x then $\partial_y \mu = 0$ and (7) reduces to the first-order linear ordinary differential equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\,\mu\,.$$

This equation will be consistent with our assumption that μ is only a function of x when the fraction on its right-hand side is independent of y. In that case you can integrate the equation to find the integrating factor

$$\mu(x) = e^{A(x)}\,, \qquad \text{where} \quad A'(x) = \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\,.$$

Similarly, when μ is only a function of y then $\partial_x \mu = 0$ and (7) reduces to the first-order linear ordinary differential equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}y} = \frac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}\,\mu\,.$$

This equation will be consistent with our assumption that μ is only a function of y when the fraction on its right-hand side is independent of x. In that case you can integrate the equation to find the integrating factor

$$\mu(y) = e^{B(y)}\,, \qquad \text{where} \quad B'(y) = \frac{\partial_x N(x,y) - \partial_y M(x,y)}{M(x,y)}\,.$$

This will be the only method for finding integrating factors that we will use in this course.

Remark: Rather than memorize the above formulas for $\mu(x)$ and $\mu(y)$ in terms of primitives, I strongly recommend that you simply follow the steps by which they were derived. Namely, you seek a μ that satisfies

$$\partial_y[M(x,y)\,\mu] = \partial_x[N(x,y)\,\mu]\,.$$

You then expand the partial derivatives as

$$M(x,y)\partial_y\mu + [\partial_y M(x,y)]\,\mu = N(x,y)\partial_x\mu + [\partial_x N(x,y)]\,\mu\,.$$

If this equation reduces to an equation that only depends on x when you set $\partial_y\mu = 0$ then there is an integrating factor $\mu(x)$. On the other hand, if this equation reduces to an equation that only depends on y when you set $\partial_x\mu = 0$ then there is an integrating factor $\mu(y)$. We will illustrate this approach with the following examples.

Example: Give an implicit general solution to the differential equation

$$(2e^x + y^3)\,\mathrm{d}x + 3y^2\,\mathrm{d}y = 0\,.$$

Solution: This differential form is not exact because

$$\partial_y(2e^x + y^3) = 3y^2 \qquad \neq \qquad \partial_x(3y^2) = 0\,.$$

You therefore seek an integrating factor μ such that

$$\partial_y[(2e^x + y^3)\mu] = \partial_x[(3y^2)\mu]\,.$$

Expanding the partial derivatives gives

$$(2e^x + y^3)\partial_y\mu + 3y^2\mu = 3y^2\partial_x\mu\,.$$

Notice that if $\partial_y \mu = 0$ then this equation reduces to $\mu = \partial_x \mu$, whereby $\mu(x) = e^x$ is an integrating factor. (See how easy that was!)

Because e^x is an integrating factor, you *know* that

$$e^x(2e^x + y^3)\,\mathrm{d}x + 3e^x y^2\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

(Of course, you should check that this is exact. If it is not then you made a mistake in finding μ!) You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = e^x(2e^x + y^3)\,, \qquad \partial_y H(x,y) = 3y^2 e^x\,.$$

By intgrating the second equation you see that $H(x,y) = y^3 e^x + h(x)$. When this expression for $H(x,y)$ is plugged into the first equation you obtain

$$y^3 e^x + h'(x) = \partial_x H(x,y) = (2e^x + y^3)e^x\,,$$

which yields $h'(x) = 2e^{2x}$. Upon taking $h(x) = e^{2x}$, the general solution satisfies

$$H(x,y) = y^3 e^x + e^{2x} = C\,.$$

In this case the general solution can be given explicitly as

$$y = (Ce^{-x} - e^x)^{\frac{1}{3}}\,,$$

where C is an arbitrary constant. □

Example: Give an implicit general solution to the differential equation

$$2xy\,\mathrm{d}x + (2x^2 - e^y)\,\mathrm{d}y = 0\,.$$

Solution: This differential form is not exact because

$$\partial_y(2xy) = 2x \qquad \neq \qquad \partial_x(2x^2 - e^y) = 4x\,.$$

You therefore seek an integrating factor μ such that

$$\partial_y[(2xy)\mu] = \partial_x[(2x^2 - e^y)\mu]\,.$$

Expanding the partial derivatives gives

$$2xy\partial_y\mu + 2x\mu = (2x^2 - e^y)\partial_x\mu + 4x\mu\,.$$

Notice that if $\partial_x\mu = 0$ then this equation reduces to $y\partial_y\mu = \mu$, whereby $\mu(y) = y$ is an integrating factor. (See how easy that was!)

Because y is an integrating factor, you *know* that

$$2xy^2\,\mathrm{d}x + y(2x^2 - e^y)\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = 2xy^2\,, \qquad \partial_y H(x,y) = 2x^2 y - ye^y\,.$$

By intgrating the first equation you see that $H(x,y) = x^2y^2 + h(y)$. When this expression for $H(x,y)$ is plugged into the second equation you obtain

$$2x^2y + h'(y) = \partial_y H(x,y) = 2x^2y - ye^y\,,$$

which yields $h'(y) = -ye^y$. Upon taking $h(y) = (1-y)e^y$, the general solution satisfies

$$H(x,y) = x^2y^2 + (1-y)e^y = C\,.$$

In this case you cannot solve for y explicitly. □

Remark: Integrating factors for the linear equations can be viewed as a special case of the foregoing method. Consider the linear first-order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = f(x)\,.$$

It can be put into the differential form

$$\big(a(x)y - f(x)\big)\,\mathrm{d}x + \mathrm{d}y = 0\,.$$

This differential form is generally not exact because when $a(x) \neq 0$ we have

$$\partial_y\big(a(x)y - f(x)\big) = a(x) \qquad \neq \qquad \partial_x 1 = 0\,.$$

You therefore seek an integrating factor μ such that

$$\partial_y\big[\big(a(x)y - f(x)\big)\,\mu\big] = \partial_x\mu\,.$$

Expanding the partial derivatives gives

$$\big(a(x)y - f(x)\big)\,\partial_y\mu + a(x)\mu = \partial_x\mu\,.$$

Notice that if $\partial_y\mu = 0$ then this equation reduces to $\partial_x\mu = a(x)\mu$, whereby an integrating factor is $\mu(x) = e^{A(x)}$ where $A'(x) = a(x)$.

Because $e^{A(x)}$ is an integrating factor, you *know* that

$$e^{A(x)}\big(a(x)y - f(x)\big)\,\mathrm{d}x + e^{A(x)}\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = e^{A(x)}\big(a(x)y - f(x)\big)\,, \qquad \partial_y H(x,y) = e^{A(x)}\,.$$

By intgrating the second equation you see that $H(x,y) = e^{A(x)}y + h(x)$. When this expression for $H(x,y)$ is plugged into the first equation you obtain

$$e^{A(x)}a(x)y + h'(x) = \partial_x H(x,y) = e^{A(x)}\big(a(x)y - f(x)\big)\,,$$

which yields $h'(x) = -e^{A(x)}f(x)$. The general solution thereby satisfies

$$H(x,y) = e^{A(x)}y - B(x) = C\,, \qquad \text{where } A'(x) = a(x) \text{ and } B'(x) = e^{A(x)}f(x)\,.$$

This is equivalent to the recipe we derived previously. □