Homework 9. Due April 29

1. **(6 points)** Consider the SDE

$$dx = b(x)dt + \sigma(x)\sqrt{\epsilon}dw, \tag{1}$$

where ϵ is a small parameter, and $\sigma(x)$ is a matrix whose determinant is nowhere zero. The Freidlin-Wentzel action for SDE (2) is given by

$$S_T(\phi) = \frac{1}{2}\int_0^T \|\dot{\phi} - b(\phi)\|^2_{A(\phi)}dt, \quad A \equiv \Sigma(x)^{-1} = (\sigma(x)\sigma(x)^T)^{-1}.$$

Here $\|\cdot\|_A$ denotes the norm associated with the symmetric positive definite matrix A: $\|y\|_A^2 := y^T Ay$. The A-inner product is given by $(y, z)_A = y^T Az$.

(a) Derive the geometric action for SDE (2) analogous to Eq. (9) in `LDT.pdf`.

(b) Use the Bellman principle of optimality to derive the direction of the minimum action path ψ' and the PDE for U similar to Eqs. (17) and (18) in `LDT.pdf`.

2. **(5 points)** Consider an SDE

$$dx = b(x)dt + \sqrt{\epsilon}dw, \quad x \in \mathbb{R}^d, \tag{2}$$

where ϵ is a small parameter. Let $\mathcal{A}$ be an attractor of the corresponding ODE $\dot{x} = b(x)$, $U(x)$ be the quasipotential with respect to $\mathcal{A}$, and $D = \{x \in \mathbb{R}^d \mid U(x) < a\}$ be a sublevel set of U where the constant a is chosen so that D lies inside the basin of $\mathcal{A}$. Assume that the invariant probability distribution in D is of the form

$$m(x) = K(x)e^{-U(x)/\epsilon}. \tag{3}$$

Plug this expression to the stationary Fokker-Planck equation. It is convenient to write it on the form

$$-\nabla \cdot J = 0, \quad \text{where} \quad J(x) := b(x)m(x) - \frac{\epsilon}{2}\nabla m(x) \tag{4}$$

is the probability current. Neglect terms of order ϵ and obtain a PDE for $K(x)$. This PDE is called the *transport equation.*

3. **(5 points)** Consider the SDE

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} b_1(x,y) \\ b_2(x,y) \end{bmatrix} dt + \sqrt{\epsilon}\begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix} \tag{5}$$

where ϵ is a small parameter. The corresponding ODE

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1(x,y) \\ b_2(x,y) \end{bmatrix} \tag{6}$$

rewritten in polar coordinates splits into two independent ODEs:

$$\frac{dr}{dt} = f(r), \tag{7}$$

$$\frac{d\theta}{dt} = g(\theta). \tag{8}$$

Suppose that f and g are smooth, $g(\theta)$ is strictly positive, and $f(r)$ has zeros at $r = 0$ and $r = r_0$, and $f(r) > 0$ for $0 < r < r_0$ and $f(r) < 0$ for $r > r_0$. Hence, the only attractor of ODE (6) is the limit cycle C at $r = r_0$.

Find the quasipotential of SDE (5) with respect to C and justify your result by showing that (i) the proposed quasipotential is indeed the infimum of the Freidlin-Wentzell action and (ii) the minimizing path from C to every point (x, y) exists. *Hint: Start with expressing b_1 and b_2 via f and g, spot an orthogonal decomposition. This decomposition gives you candidates for the potential and rotational components. Integrate the candidate for the potential component and verify that the result is indeed the quasipotential.*