Homework 1. Due Wednesday, Feb. 13

1. (**5pts**) Consider a particle in 1D in contact with a heat bath whose states follow the canonical distribution:
$$\mu(x,p) = \frac{1}{Z}e^{-\beta H(x,p)}, \quad \text{where} \quad Z = \int_{\mathbb{R}^2} e^{-\beta H(x,p)}dxdp, \tag{1}$$
where $H(x,p) = V(x) + \frac{p^2}{2}$ is its energy and $\beta = (k_B T)^{-1}$ (k_B is Boltzmann's constant).
   (a) Show that the mean kinetic energy equals to $k_B T/2$, i.e., calculate the expected value of
$$E\left[\frac{p^2}{2}\right] = \frac{1}{Z}\int_{\mathbb{R}^2} \frac{p^2}{2}e^{-\beta(V(x)+p^2/2)}dxdp.$$
   (b) Use your result to show that for a system consisting of n particles of unit mass each of which is moving in 3D, the mean kinetic energy is $(3/2)nk_B T$. The canonical distribution of this system is given by
$$\mu(x,p) = \frac{1}{Z}e^{-\beta H(x,p)}, \quad x,p \in \mathbb{R}^{3n}, \quad Z = \int_{\mathbb{R}^{6n}} e^{-\beta H(x,p)}dxdp, \tag{2}$$
$$H(x,p) = V(x) + \frac{1}{2}\sum_{i=1}^{3n} p_i^2.$$

2. (**5pts**) Suppose you are throwing two dice. Consider the random variables $\eta = \omega_1 + \omega_2$ (the sum of numbers on the top) and $\theta = |\omega_1 - \omega_2|$ (the absolute value of the difference of the numbers on the top).
   (a) Determine whether η and θ are dependent.
   (b) Calculate $\text{Cov}(\eta, \theta)$.
   (c) Calculate $E[\eta|\theta]$ and $\text{Var}(\eta|\theta)$.
   (d) Calculate $E[\theta|\eta]$ and $\text{Var}(\theta|\eta)$.

3. (**5pts**) Problem 9 from Chorin & Hald, 2nd edition [1], Chapter 2, p. 45. (Matches problem 4, Chapter 2 in the 3rd edition of Chorin & Hald (2013).)

4. (**5pts**) Problem 10 from Chorin & Hald, 2nd edition [1], Chapter 2, p. 45. (Matches problem 5, Chapter 2 in the 3rd edition of Chorin & Hald (2013).)

References

[1] A. Chorin and O. Hald, Stochastic Tools in Mathematics and Science, 2nd edition, Springer 2009. You can download a pdf file with the whole book from the UMD library.