Homework 3. Due Feb. 27

1. **(5 pts)**

   (a) Suppose you have n independent samples $x_1, \ldots, x_n$ of a random variable η. Prove that the estimate for the variance

   $$S_2^2 := \frac{1}{n-1}\sum_{k=1}^{n}(x_k - m)^2, \quad \text{where} \quad m = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

   is unbiased.

   (b) An exponential random variable η with parameter λ has pdf $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$. Suppose that the parameter λ is unknown, but you have n independent samples $x_1, \ldots, x_n$ of η. What is the maximum likelihood estimate for λ?

2. **(5 pts)** (This problem is based on an example from [1].) Consider the integral

   $$I = \int_0^1 \cos(x/5)e^{-5x}dx.$$

   The exact value of I is

   $$\frac{1}{626}\left(125 - 125e^{-5}\cos(1/5) + 5e^{-5}\sin(1/5)\right).$$

   (a) Evaluate I by Monte-Carlo (MC) as $I = E[\cos(\eta/5)e^{-5\eta}]$, where η is a random variable uniformly distributed on $[0, 1]$. Make your code estimate the standard deviation $\sqrt{\mathrm{Var}(\cos(\eta/5)e^{-5\eta})}$. Estimate error of your MC result. Estimate the numbers of samples required to achieve relative errors of 1% and 0.1%.

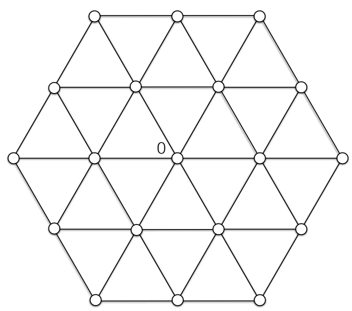
   (b) Evaluate I by Monte-Carlo using importance sampling, i.e, as $I = I_1 E[\cos(\xi/5)]$, where ξ is a random variable with the pdf

   $$f_\xi(x) = \begin{cases} I_1^{-1}e^{-5x} & , x \in [0,1] \\ 0, & x \notin [0,1], \end{cases}$$

   where $I_1 = \int_0^1 e^{-5x}dx$. Make your code estimate the standard deviation $\sqrt{\mathrm{Var}(\cos(\xi/5))}$. Estimate error of your MC result. Estimate the numbers of samples required to achieve relative errors of 1% and 0.1%.

3. **(5 pts)** Consider the Markov chain associated with the graph shown in the Figure below. For any vertex i, $P_{ij} > 0$ iff i and j are connected by an edge, and $P_{ij} = 1/d(i)$, where $d(i)$ is the number of edges emanating from i (the degree of i). Denote the

state at the center by 0. Find the expected hitting times $k_i^{\{0\}}$ to hit 0 from each state $i \neq 0$. *Hint: using the symmetry of the problem you can dramatically decrease the number of equations in the system to be solved.*



4. **(5 pts)** Consider the discrete time Markov chain with the infinite set of states $S = \{0, 1, 2, \ldots\}$ and the transition matrix P such that $P_{i,i-1} = q$, $P_{i,i+1} = p$, $i = 1, 2, \ldots$, where $p + q = 1$, and all other entries are zero. Note that 0 is absorbing.

   (a) Assume that $p = q = 1/2$. Show that the expected hitting times $k_i^{\{0\}} = \infty$ to hit 0 for all $i \geq 1$.

   (b) Assume that $q > p$. Find the expected hitting times k_i^0 for $i \geq 1$.

References

[1] A. Chorin and O. Hald, Stochastic Tools in Mathematics and Science, 2nd edition, Springer 2009