Homework 5. Due March 13

1. **(10 pts)** The goal of this problem is to get an exposure to the application of Monte Carlo simulations to social dynamics.

   (a) Read the description of the Deffuant model for opinion dynamics in [1] (pages 18 – 20).

   (b) Consider the Deffuant model on a Erdos-Renyi random graph $G(N, p)$, where N is the number of vertices, and each pair of vertices is connected by an edge with probability p. You are welcome to use my code `MakeRandomGraph.m` to generate a such a random graph, or write your own code. To ensure that the resulting random graph is connected, you can test it using e.g. the standard algorithm for graph exploration called the Depth-First-Search (DFS) (it is implemented in my code).

   (c) The goal of our study of the Deffuant model on Erdos-Renyi random graphs is to understand how the number of the formed opinion clusters depends on the parameter p in the Erdos-Renyi random graph and on the parameter ϵ in the Deffuant model. Set the convergence parameter $\mu = 0.3$ in the Deffuant model, and the number of vertices $N = 100$ in the random graph. Use the values of ϵ from 0.05 to 0.3 with step 0.05 in the Deffuant model, and the values $p = 0.1$, 0.5 and 0.9 in the Erdos-Renyi random graph.

   For each pair of values (ϵ, p), generate 100 random graphs. For each random graph, generate 100 initial distributions of opinions sampled from the uniform distribution on $[0, 1]$. For each initial setup, run the Deffuant model for $300N$ time steps. (Feel free to reduce these numbers if your runtimes are too long, but if you reduce the number of time steps, make sure that the formation of clusters is completed by the end of your computation.) Then determine the number of opinion clusters. I consider opinions lying within the same cluster of the difference between them does not exceed 10^{-3}. Compute the mean number of clusters and the standard deviation for it for each pair (ϵ, p). In MATLAB, you can use the standard functions `mean` and `var` (`std = sqrt(var(Nclusters))`.

   (d) Submit the following items:

      i. a printout of the mean numbers of clusters and the corresponding standard deviations for each pair of values (ϵ, p);

      ii. plots similar to the one in Fig. 6 in [1] for $p = 0.1$ and $p = 0.9$ for $\epsilon = 0.05,\ 0.15,\ 0.25$ for one random graph and one initial distribution on it (6 plots in total);

iii. a summary of your observations. (Do the mean numbers of clusters approach some limits as $p \to 1$? What happens to the standard deviations as $p \to 1$? Compare the results for different values of p.)

2. **(5 pts)** Prove that the frequencies of visits of states in the Metropolis-Hastings algorithm approach the invariant distribution.

3. **(5 pts)** A random variable $\eta : \Omega \to [0, \infty]$ has an exponential distribution if

$$\mathbb{P}(\eta > t) = e^{-\lambda t} \quad \text{for all} \quad t \geq 0,$$

where $\lambda \geq 0$ is a parameter.

Prove that a random variable $\eta : \Omega \to (0, \infty]$ has an exponential distribution if and only if it has the following memoryless property:

$$\mathbb{P}(\eta > t + s \mid \eta > s) = \mathbb{P}(\eta > t) \quad \text{for all} \quad s, t \geq 0. \tag{1}$$

Hint: To deduce an exponential distribution from Eq. (1), you may proceed as follows. Introduce

$$F(t) := \mathbb{P}(\eta > t) \quad \textit{and define} \quad \lambda := -\log F(1).$$

Then write $F(1) = F\left(\frac{1}{n} + \ldots + \frac{1}{n}\right)$ *and apply the memoryless property. Establish the exponential form for* F *for all rational* t*. Then use the fact that any real number can be approximated by rational numbers. Etc.*

References

[1] Claudio Castellano, Santo Fortunato, Vittorio Loreto, Statistical physics of social dynamics, Rev. Mod. Phys. 81, 591 (2009), arXiv:0710.3256