Homework 8. Due April 17

1. **(5 pts)** Consider the mesh

$$\{t_j \mid t_j = jh, \quad h = \tfrac{1}{N}, \quad 0 \le j \le N\}.$$

Let $\{z_j\}_{j=1}^N$ be independent Gaussian random variables with mean 0 and variance 1. Consider the Gaussian random walk $B_h(t)$ defined by

$$B_h(0) = 0,$$
$$B_h(t_j) = B_h(t_{j-1}) + z_j\sqrt{h}, \quad j = 1, \ldots, N,$$
$$B_h(t) = \frac{1}{h}\left[B_h(t_{j-1})(t_j - t) + B_h(t_j)(t - t_{j-1})\right], \quad t_{j-1} < t < t_j, \quad j = 1, \ldots, N.$$

Prove that this random walk satisfies axioms (1)-(4) of Brownian motion at the points t_j, $j = 0, 1, \ldots, N$.

2. **(5 pts)** Consider the construction of a Brownian motion via the refinement procedure described in Section 1.1.1 in `SDEs.pdf`. Verify that for any $n = 0, 1, 2, \ldots$, B_n satisfies axioms (1)-(4) of Brownian motion at the dyadic points.

3. **(5 pts)** Problem 3, Chapter 3, page 78, from Chorin&Hald, 2nd edition [1]. Evaluate exactly $\int F dW$ for the following functionals F:

   (a) $F = \sin(w^3(1))$;

   (b) $F = \sin\left(w^2\left(1/2\right)\right)\cos(w(1))$;

   (c) $F = \int_0^{1/2} w^2(s) w^2\left(1/2 + s\right) ds$.

References

[1] A. Chorin and O. Hald, Stochastic Tools in Mathematics and Science, 2nd edition, Springer 2009