Homework 9. Due April 24

1. **(9 pts)** Consider a stochastic process $f(t,\omega)$ on $0 \le t \le T$ where ω indicates that f depends on a Brownian motion. Assume that:

   (a) $f(t,\omega)$ is independent of the increments of the Brownian motion $w(t,\omega)$ in the future, i.e., $f(t,\omega)$ is independent of $w(t+s,\omega) - w(t,\omega)$ for all $s > 0$.

   (b)
$$\int_0^T E[f^2(s,\omega)]ds < \infty.$$

   Derive the properties below from the definition of the Ito stochastic integral (page 14 in `SDEs.pdf`).

   (a) If f is a deterministic function, i.e., $f(s,\omega) \equiv f(s)$, then
$$\int_0^t f(s)dw(s,\omega) \sim N\left(0, \int_0^t f^2(s)ds\right).$$

   (b) For any $0 \le \tau \le t \le T$,
$$E\left[\int_\tau^t f(s,\omega)dw(s,\omega)\right] = 0;$$

   (c) For any $0 \le \tau \le t \le T$,
$$E\left[\int_\tau^t f(s,\omega)dw(s,\omega)\int_\tau^t g(s,\omega)dw(s,\omega)\right] = \int_\tau^t E[f(s,\omega)g(s,\omega)]ds.$$

2. **(6 pts)** Consider the SDE
$$dX_t = b(X_t)dt + \sigma(X_t)dw, \quad X(0) = x \in \mathbb{R}^d, \quad t \in [0,T]. \tag{1}$$

   Show that
$$\lim_{t\to s} E\left[\frac{X_t - X_s}{t-s} \mid X_s = x\right] = b(x,s) \tag{2}$$
$$\lim_{t\to s} E\left[\frac{[X_t - X_s][X_t - X_s]^T}{t-s} \mid X_s = x\right] = \Sigma(x,s). \tag{3}$$

   The vector field $b(x,s)$ is called drift, and the matrix $\Sigma(x,s) = \sigma(x,s)\sigma(x,s)^T$ is called the diffusion matrix.

3. **(5 pts)** Find the analytical solution of the initial value problem
$$dX_t = \left(\frac{b^2}{4} - X_t\right)dt + b\sqrt{X_t}dw, \quad X_0 = x > 0,$$

   where b is constant. Note that this process will stop as X_t reaches 0.

   Hint: make the variable change $Y = \sqrt{X}$ using the Ito formula.