EXPOSITORY NOTES ON DISTRIBUTION THEORY, AMSC/MATH 673, FALL 2018

While these notes are under construction, I expect there will be many typos.

The main reference for this is volume 1 of Hörmander, The analysis of liner partial differential equations. I have picked a few of the most useful and concrete highlights. The references are based on the 1989 hardcover second edition.

1. Generalities (from Ch. 2 and 3)

Definition 1.1. Let U be an open set in $\mathbb{R}^n$. A distribution $u \in \mathcal{D}'(U)$ is a linear function $u : C_0^\infty(U) \to \mathbb{C}$. One can write $u(\phi) =< u, \phi >$ and think of this, informally, as $u(\phi) = \int u\phi$. It is required that u is continuous in the following sense:

For every $K \subset U$ compact there exist C, k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_x |\partial^\alpha \phi| \tag{1}$$

for every $\phi \in C_0^\infty(U)$ supported in K.

If one k works for all K, u is of finite order. The smallest such k is the order of u.

We will need an equivalent formulation of the continuity condition.

Definition 1.2. Let $\phi_j, \phi \in C_0^\infty(U)$. The sequence $\phi_j \to \phi$ in $C_0^\infty(U)$ if there exists a compact subset of U which contains the support of all ϕ_j, ϕ and for every fixed α, $\sup_x |\partial^\alpha (\phi_j(x) - \phi(x))| \to 0$ as $j \to \infty$.

Theorem 1.3. *A linear function $u : C_0^\infty(U) \to \mathbb{C}$ is a distribution if and only if $u(\phi_j) \to u(\phi)$ for every $\phi_j \to \phi$ in $C_0^\infty(U)$.*

Proof. To show that if u is a distribution, then $u(\phi_j) \to u(\phi)$ for every $\phi_j \to \phi$ in $C_0^\infty(U)$ is clear from the definition. The other half is an easy exercise in negations. □

Examples:

(1) If $\tilde{u}$ is a locally integrable function, $u(\phi) := \int \tilde{u}\phi$. This identifies the function $\tilde{u}$ with a distribution u.
(2) Dirac delta function. $\delta_a(\phi) = \phi(a)$

(3) Weak derivatives: If u is a locally integrable function, $< \partial^\alpha u, \phi >:= (-1)^{|\alpha|} \int u \partial^\alpha \phi$. This agrees with integration by parts if u is a smooth function and is in fact the definition of $\partial^\alpha u$ for any distribution u: $\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi)$.

(4) It takes some work (thm. 4.4.7 in Hörmander) and we will not prove this, but the above essentially accounts for all possible distributions:

If $u \in \mathcal{D}'(U)$ then there exists a locally finite family of continuous functions f_α (each compact subset of U intersects only finitely many of the supports of the f_αs) such that

$$u(\phi) = \sum \int f_\alpha \partial^\alpha \phi$$

Definition 1.4. A sequence of distributions u_i converges to u in $\mathcal{D}'(U)$ (or in the sense of distribution theory) if $u_i(\phi) \to u(\phi)$ for every $\phi \in C_0^\infty(U)$

Also, if $u_i \in \mathcal{D}'(U)$ and for each fixed $\phi \in C_0^\infty(U)$ the limit $u_i(\phi)$ exists and is denoted $u(\phi)$, then u is automatically a distribution. See Theorem 2.1.8. We will not prove this.

Definition 1.5. Let $u \in \mathcal{D}(U)$ and $f \in C^\infty(U)$. Then the distributions $\frac{\partial u}{\partial x_k}$ and fu are defined by

$$\left(\frac{\partial u}{\partial x_k}\right)(\phi) = -u\left(\frac{\partial}{\partial x_k}\right)$$

$$(fu)(\phi) = u(f\phi)$$

Unlike classical convergence, if $u_i \to u$ in $\mathcal{D}'(U)$, then $\partial^\alpha u_i \to \partial^\alpha u$ in $\mathcal{D}'(U)$ is trivial.

Example 1: Let H be the Heavyside function. Then $H' = \delta_0$.

The following two propositions will be proved in class. (See Chapter 3 in Hörmander)

Proposition 1.6. *Let u be continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus x_0$. Let $v = u'$ on $\mathbb{R} \setminus x_0$, and assume the function v is locally integrable. Then $u' = v$ in the sense of distribution theory.*

Proposition 1.7. *Let $u \in \mathcal{D}'(\mathbb{R})$, and assume $u' = 0$. Then u is constant.*

Remark that if u is the Cantor function, u is continuous and $u' = 0$ a.e, but $u' \neq 0$ in the sense of distribution theory.

More examples to be worked out in class:

- If E is the fundamental solution of the Laplace operator, ∇E in the sense of distributions agrees with the locally integrable function ∇E defined for $x \neq 0$, but ΔE in the sense of distributions does not agree with the locally integrable function $\Delta E = 0$ defined for $x \neq 0$. In fact $\Delta E = \delta_0$.
- $\frac{\partial f}{\partial \bar{z}} \frac{1}{z} = \frac{1}{\pi}\delta_0$. This follows from the fundamental solution of the Laplace operator in the plane.
- Let f analytic in $\mathbb{C} \setminus \{0\}$, and assume f is bounded in a neighborhood of 0. Then $\frac{\partial f}{\partial \bar{z}} = 0$ in the sense of distribution theory. We will see later that this implies f analytic, thus f has a removable singularity.
- Let u continuous in the plane, and harmonic for $y > 0$ Assume u extends as a C^1 (but possibly not C^2)function to $y \geq 0$. Assume $u(x, 0) = 0$, and $u(x, y) = -u(x, -y)$. Then $\Delta u = 0$ in the plane (weakly). We will see that this implies u is harmonic classically. This is why Schwarz reflection works.

Definition 1.8. A distribution u is defined to be 0 in an open set $V \subset U$ if $u(\phi) = 0$ for every $\phi \in C_0^\infty(V)$. The union of all such subsets V is the largest open set where u is 0, and the complement of that is defined to be the support of u.

Thus the support of a distribution $u \in \mathcal{D}(U)$ is always (relatively) closed in U. If the support of u is compact, u is called compactly supported. The set of compactly supported distributions in U is denoted by $\mathcal{E}'(U)$

Recall the support of a function ϕ is the closure of the set $\{\phi(x) \neq 0\}$. If $u \in R^n$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, and the support of ϕ and u are disjoint, then $u(\phi) = 0$. However, if ϕ is zero on the support of u, it does not follow that $u(\phi) = 0$. Example: $\delta'(x)$.

If $u \in \mathcal{E}'(U)$, $u(\phi)$ is well defined for $\phi \in C^\infty$: Let K be the support of u, $K \subset V \subset U$ with V open. There exists a smooth cut-off function $\zeta \in C_0^\infty(U)$, and $\zeta = 1$ in V. Then $u(\zeta\phi)$ is well-defined, and is independent of the choice of ζ. Define $u(\phi) = u(\zeta\phi)$ for ζ as above.

Definition 1.9. A distribution u is defined to be smooth in an open set $V \subset U$ if there exists $\tilde{u} \in C^\infty(V)$ such that $u(\phi) = \int \tilde{u}(x)\phi(x)dx$ for all $\phi \in C_0^\infty(V)$ The union of all such subsets V is the largest open set where u is smooth, and the complement of that is defined to be the singular support of u.

The major goals of this sections of the course are to prove

1) If $f \in \mathcal{E}'$, then there exists $u \in \mathcal{D}'$ such that $\Delta u = f$.

2) If u, f are as above, and $f \in C^\infty(V)$ for some open set V, then $u \in C^\infty(V)$.

Both of these goals follow from the properties of the convolution of a distribution with a compactly supported distribution. Part 1 follows by writing $u = E * f$, $\Delta u = (\Delta E) * f = \delta * f = f$, but we have to assign rigorous meaning to this. Part 2 follows from the fact that the fundamental solution E is C^∞ away from 0. The exact same results hold for $\frac{\partial}{\partial t} - \Delta$ and $\frac{\partial}{\partial \bar{z}}$, but not $\frac{\partial^2}{\partial t^2} - \Delta$.

First, two short digressions.

2. Distributions supported at one point

Theorem 2.1. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ is supported at a point, say 0, then u is a finite linear combination*

$$u = \sum c_\alpha \partial^\alpha \delta$$

Proof. Assume u is of order k (and prove: any compactly supported distribution is of finite order). Pick a test function ϕ and write $\phi(x) = T(x) + R(x)$ the kth order Taylor polynomial plus remainder. $u(T)$ is what we want (check!), and the point is to show that $u(R) = 0$ where R is the remainder. We know $|R(x)| \le C|x|^{k+1}$ for $|x| \le 1$ and in fact $|\partial^\alpha R(x)| \le C|x|^{k+1-|\alpha|}$ for all $|\alpha| \le k$. Let $\epsilon > 0$, and let χ be a cut-off function, identically 1 in a neighborhood of 0.
Then $|u(R)| = |u(\chi\left(\frac{x}{\epsilon}\right)R\big) \le C\sum_{|\alpha|\le k} \sup_x |\partial^\alpha(\chi(\frac{x}{\epsilon})R)| \le C\epsilon$. Now let $\epsilon \to 0$. ☐

Application to PDE: Let $E = \frac{1}{|x|^{n-2}}$ $(n \ge 3)$. Then $\Delta E = 0$ for x away from 0 by calculation, thus ΔE is a distribution supported at 0. It is a finite linear combination of the delta function and its derivatives. An additional homogeneity argument shows $\Delta E = c\delta$.

If u is a locally integrable function in $\mathbb{R}^n - \{0\}$, u is homogeneous of degree α if $u(tx) = t^\alpha u(x)$ for all $t > 0$ and $x \ne 0$. Denoting $\phi_t(x) = t^n \phi(tx)$ this is equivalent to

$$\int u\phi = t^\alpha \int u\phi_t$$

and the definition of a homogeneous distribution in $\mathbb{R}^n$ (or $\mathbb{R}^n - \{0\}$) is

$$u(\phi) = t^\alpha u(\phi_t)$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$ or $C_0^\infty(\mathbb{R}^n - \{0\})$.

3. The gradient of a characteristic function and homogeneous distributions

Our next theorem (formula 3.1.5 in Hörmander's book) is

Theorem 3.1. *Let U be an open set with C^1 boundary. Then*

$$\nabla \chi_U = -\nu dS$$

where ν is the outward pointing normal.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a smoothed out Heaviside function: $h(x) = 0$ if $x \leq 0$, $h(x) = 1$ if $x \geq 1$ and smooth in-between. It suffices the prove the theorem for test functions ϕ supported in a small neighborhood of $x_0 \in \partial U$, where U agrees with $x_n > r(x_1, \cdots x_{n-1})$. Then

$$\int \chi_U \phi = \lim_{\epsilon \to 0} \int h(\frac{x_n - r(x_1, \cdots x_{n-1})}{\epsilon}) \phi(x_1, \cdots, x_n)$$

by the Lebesgue dominated convergence theorem, and

$$\nabla \chi_U(\phi) = \lim_{\epsilon \to 0} \int \nabla \left(h(\frac{x_n - r(x_1, \cdots x_{n-1})}{\epsilon}) \right) \phi(x_1, \cdots, x_n)$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \frac{1}{\epsilon} h'(\frac{x_n - r(x_1, \cdots x_{n-1})}{\epsilon}) \cdot (-\nabla r(x_1, \cdots, x_{n-1}), 1) \phi(x) dx$$

$$= \int_{\mathbb{R}^{n-1}} \phi(x_1, \cdots x_{n-1}, r(x_1, \cdots r_{x-1})) \cdot (-\nabla r(x_1, \cdots, x_{n-1}), 1) \phi dx_1 \cdots dx_{n-1}$$

integrate x_n first

$$= -\int_{\partial U} \phi \cdot \nu \phi dS$$

(by the Calculus formulas for ν and dS). We used the fact that $\frac{1}{\epsilon} h'(\frac{x}{\epsilon})$ is an "approximation to the identity". We will see another proof of this important fact in the chapter on compositions with smooth functions. □

4. Convolutions (Chapter 4 in Hörmander's book)

Definition 4.1. If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$,

$u * \phi(x) = u(\phi(x - \cdot))$ (where $\cdot$ stands for y, and u acts in the y variable)

Check $u * \phi \in C^\infty$, $\partial^\alpha (u * \phi)(x) = (\partial^\alpha u) * \phi = u * (\partial^\alpha \phi)(x)$: We have

$$\phi(x - y + \epsilon e_i) - \phi(x - y) = \frac{\partial}{\partial x_i} \phi(x - y) + R(x - y, \epsilon)$$

where

$$R(x-y,\epsilon) = \int_0^1 \frac{d^2}{dt^2}\left(\phi(x-y+t\epsilon e_i)\right)(1-t)dt$$
$$= \epsilon^2 \int_0^1 \left(\frac{\partial^2\phi}{\partial x_i^2}\right)(x-y+t\epsilon e_i)(1-t)dt$$

Fix x. $R(x-y,\epsilon)$ is in C_0^∞, and $\sup_y |\partial_y^\alpha R(x-y,\epsilon)| \le C_\alpha \epsilon^2$. Using the continuity condition (1) we see

$$\lim_{\epsilon\to 0} \frac{R(x-\cdot,\epsilon)}{\epsilon} = 0$$

and

$$\lim_{\epsilon\to 0} \frac{u\left(\phi(x-\cdot+\epsilon e_i)\right) - u\left(\phi(x-\cdot)\right)}{\epsilon} = u(\frac{\partial}{\partial x_i}\phi(x-\cdot)) = \frac{\partial u}{\partial x_i}(\phi(x-\cdot)$$

Check *support*$(u*\phi) \subset$ *support* u + *support* ϕ: Fix x. If $\phi(x-\cdot)$ is supported in the complement of *support* u, then $u(\phi(x-\cdot) = 0$ by the definition of *support* u. If $u(\phi(x-\cdot) \neq 0$, then $\exists\ y \in$ *support* u and $y \in$ *support* $\phi(x-\cdot)$. Thus $y = \lim y_i$ with $\phi(x-y_i) \neq 0$, and

$$x = \lim\,(x-y_i+y) \in \overline{support\,\phi + support\,u} = support\,\phi + support\,u$$

because *support* u is compact.

We also have

Theorem 4.2. *Let* $u \in \mathcal{D}'(\mathbb{R}^n)$*, and* $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$*. Then* $(u*\phi)*\psi = u*(\phi*\psi)$.

Proof. Before starting the proof, review Definition (1.2). $u*\phi \in C^\infty$. Fix x.

$$(u*\phi)*\psi(x) = \int (u*\phi)(x-y)\psi(y)dy$$
$$= \lim_{h\to 0+} \sum_{k\in\mathbb{Z}^n} (u*\phi)(x-kh)\psi(kh)h^n$$
$$= \lim_{h\to 0+} u\left(\sum_{k\in\mathbb{Z}^n} \phi(x-kh-\cdot)\psi(kh)h^n\right)$$
$$= u\left(\int \phi(x-z-\cdot)\psi(z)dz\right)$$

In the last line, we used the (obvious) fact that, for x fixed,

$$\sum_{k\in\mathbb{Z}^n} \phi(x-kh-y)\psi(kh)h^n \to \int \phi(x-z-y)\psi(z)dz$$

uniformly in y, and the same is true for after differentiating with respect to y an arbitrary number of times. Also, both LHS and RHS are supported in a fixed compact set. In other words, LHS $\to$ RHS in C_0^∞. □

This implies the important theorem on approximating distributions by C^∞ functions.

Theorem 4.3. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$, and let η_ϵ be the standard mollifier. Then $u * \eta_\epsilon \in C^\infty(\mathbb{R}^n)$ and $u * \eta_\epsilon \to u$ in the sense of distribution theory (as $\epsilon \to 0$).*

Proof. We have to check

$$(u * \eta_\epsilon)(\phi) \to u(\phi)$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$. The proof is based on the observation that $u(\phi) = u * \phi_-(0)$ where $\phi_-(x) = \phi(-x)$. So it suffices to show $(u * \eta_\epsilon) * \phi(0) \to u * \phi(0)$. But

$$(u * \eta_\epsilon) * \phi(0) = u * (\eta_\epsilon * \phi)(0) \to u * \phi(0)$$

since $\eta_\epsilon * \phi \to \phi$ in C_0^∞. □

Now we define the convolution of two distribution u_1, u_2, one of which is compactly supported.

This is defined so that the formula

$$(u_1 * u_2) * \phi = u_1 * (u_2 * \phi)$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$. For simplicity, let's assume u_2 is compactly supported. Instead of defining $(u_1 * u_2)(\phi)$ it suffices to define $(u_1 * u_2) * \phi(0)$. This is done in the obvious way:

$$(u_1 * u_2) * \phi(0) = u_1 * (u_2 * \phi)(0)$$

We have to check that $u_1 * u_2$ satisfies the continuity condition. Let $\phi_j \to 0$ in $C_0^\infty(\mathbb{R}^n)$ (see Definition (1.2)). Then so does $u_2 * \phi_j$, and $u_1 * (u_2 * \phi_j)(0) \to 0$.

Also, it τ_h denotes a translation, $(\tau_h \phi)(x) = \phi(x+h)$, then $\tau_h(u*\phi) = u * (\tau_h \phi)$ and

$$\begin{aligned}(u_1 * u_2) * \phi(h) = \tau_h\left((u_1 * u_2) * \phi\right)(0) &= \left((u_1 * u_2) * \tau_h \phi\right)(0) \\ = u_1 * (u_2 * \tau_h \phi)(0) &= u_1 * \left(\tau_h(u_2 * \phi)\right)(0) \\ &= u_1 * (u_2 * \phi)(h)\end{aligned}$$

Proposition 4.4. *Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$, one of which is compactly supported. Then*

$$\textit{support}\,(u_1 * u_2) \subset \textit{support}\, u_1 + \textit{support}\, u_2$$

Proof. Let η_ϵ be a standard mollifier supported in a ball or radius ϵ. It suffices to show

$$support\,(u_1 * u_2) \subset support\,u_1 + support\,u_2 + support\,\eta_{\epsilon_0}$$

for all $\epsilon_0 > 0$. We do know

$$\begin{aligned} support\,(u_1 * u_2 * \eta_\epsilon) &\subset support\,u_1 + support\,u_2 + support\,\eta_\epsilon \\ &\subset support\,u_1 + support\,u_2 + support\,\eta_{\epsilon_0} \end{aligned}$$

for all $0 < \epsilon < \epsilon_0$. Also remark that if A is closed and u is a distribution such that $support\,u * \eta_\epsilon \subset A$ for all $\epsilon_0 > \epsilon > 0$, then $support\,u \subset A$. This amounts to showing that if $u * \eta_\epsilon = 0$ in A^c, then $u = 0$ in A^c, which follows from $u * \eta_\epsilon \to u$ in the sense of distributions. □

Theorem 4.5. *Let u_1, u_2, u_3 distributions in $\mathbb{R}^n$, two of which are compactly supported. Then*

$$(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$$

Proof. The proof follows by noticing it suffices to check $((u_1 * u_2) * u_3) * \phi = (u_1 * (u_2 * u_3)) * \phi$ for every $\phi \in C_0^\infty(\mathbb{R}^n)$ which follows easily from the defining property of Theorem (4.2). □

Theorem 4.6. *Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$, one of which is compactly supported. Then*

$$u_1 * u_2 = u_2 * u_1$$

Proof. The strategy is to show that $(u_1 * u_2) * (\phi * \psi) = (u_2 * u_1) * (\phi * \psi)$ for all test functions ϕ, ψ. This is done using the associativity property Theorem (4.2) together with the fact that convolutions of functions is commutative. We will not prove this □

Theorem 4.7. *Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$, one of which is compactly supported. Then*

$$\partial^\alpha(u_1 * u_2) = (\partial^\alpha u_1) * u_2 = u_1 * \partial^\alpha u_2 \tag{2}$$

Proof. We already know $\partial^\alpha(u * \phi) = (\partial^\alpha u) * \phi = u * (\partial^\alpha \phi)$, so the theorem is proved by convolving (2) with ϕ. □

Theorem 4.8. *Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$, one of which is compactly supported. Then*

$$sing\ support\,(u_1 * u_2) \subset sing\ support\,u_1 + sing\ support\,u_2$$

Proof. The proof is based on the fact that if one of u_1, u_2 is smooth, so is $u_1 * u_2$. Let χ_1, χ_2 be supported in small neighborhoods of $sing\ support\,u_1$, $sing\ support\,u_2$, so that $(1 - \chi_1)u_1$ and $(1 - \chi_2)u_2$ are smooth. Then

$$sing\ support\,(u_1 * u_2) \subset sing\ support\,(\chi_1 u_1) * (\chi_2 u_2) \subset support\,\chi_1 u_1 + support\,\chi_2 u_2$$

□

Now we come back to PDEs. Let $P(D)$ be a constant coefficient differential operator. A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution if $P(D)E = \delta$. We already know formulas for (the) fundamental solution of the Laplace and heat operators. We will write down later several fundamental solutions of the wave operator.

Theorem 4.9. *If sing support $(E) = \{0\}$, U is open and $u \in \mathcal{D}'(U)$ is such that $P(D)u \in C^\infty(U)$, then $u \in C^\infty(U)$*

Proof. Let $V \subset\subset U$ an arbitrary open subset. It suffices to show $u \in C^\infty(V)$. Let $\zeta \in C_0^\infty(U)$, $\zeta = 1$ on V. Then $P(D)(\zeta u) = P(D)u$ in V, and in particular is C^∞ there. Finally,

$$\zeta u = \zeta u * \delta = \zeta u * P(D)(E) = (P(D)(\zeta u)) * E$$

and therefore

$$\textit{sing support}\,(\zeta u) \subset \textit{sing support}\,(P(D)(\zeta u)) + \{0\} = \textit{sing support}\,(P(D)(\zeta u))$$

But we know that *sing support* $(P(D)(\zeta u))$ is disjoint from V, so *sing support* (ζu) is also disjoint from V, in other words ζu, which equals u in V, is smooth there. □

5. The Fourier transform

Definition 5.1. The space of Schwartz functions $\mathcal{S}$ is defined by the requirement that all semi-norms

$$\sup_x |x^\alpha \partial^\beta f|$$

be finite. Convergence in this space means

$$\sup_x |x^\alpha \partial^\beta (f_n - f)| \to 0$$

for all α, β.

The Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

The following are elementary properties which will be checked in class:

Lemma 5.2. *Let* $f \in \mathcal{S}$*, denote* $f_\lambda(x) = f(\lambda x)$ $(\lambda > 0)$*,* $\tau_y f(x) = f(x+y)$ $(y \in \mathbb{R}^n$ $)$ *and* $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$*. Then* $\hat{f} \in \mathcal{S}$ *and* $f \to \hat{f}$ *is continuous in the topology of* $\mathcal{S}$*. Also,*

$$\hat{f}_\lambda(\xi) = \frac{1}{\lambda^n}\hat{f}(\frac{\xi}{\lambda})$$
$$\mathcal{F}(\tau_y f)(\xi) = e^{ix\cdot\xi}\hat{f}(\xi)$$
$$\mathcal{F}(D_j f)(\xi) = \xi_j \hat{f}(\xi)$$
$$\mathcal{F}(x_j f)(\xi) = -D_j\hat{f}(\xi)$$
$$\mathcal{F}\left(e^{-\frac{|x|^2}{2}}\right)(\xi) = (2\pi)^{n/2}e^{-\frac{|\xi|^2}{2}}$$
$$\int f\hat{h} = \int \hat{f}g \quad \textit{for all } \hat{f},\, \hat{h} \in \mathcal{S}$$

These easily imply the inversion formula and Plancherel formulas , which will be proved in class.

Theorem 5.3. *Let* $f \in \mathcal{S}$*. Then*

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)d\xi$$

Also,

$$\int_{\mathbb{R}^n} f(x)\overline{g}(x)dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\hat{g}}(\xi)d\xi$$

Definition 5.4. The space of continuous linear functionals $u : \mathcal{S} \to \mathbb{C}$ is the space of tempered distributions $\mathcal{S}'$. $u \in \mathcal{S}'$ if and only if there exists N and C such that

$$| < u, \phi > | \le C \sum_{|\alpha|,|\beta|\le N} \sup_x |x^\alpha \partial^\beta(f)|$$

for all $\phi \in \mathcal{S}$. If $u \in \mathcal{S}'$, then $\hat{u} \in \mathcal{S}'$ is defined by

$$< \hat{u}, \phi > = < u, \hat{\phi} >$$

for all $\phi \in \mathcal{S}$.

Example: The constant function $1 \in \mathcal{S}$ and $\hat{1} = (2\pi)^n\delta$.

6. Fractional H^s spaces and the sharp trace theorem

Let $u \in \mathcal{S}$. Since $\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\|\xi^\alpha\hat{u}\|^2_{L^2(\mathbb{R}^n)}$, there exist constants $c > 0$, $C > 0$ such that

$$c\sum_{\alpha=k}\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} \le \||\xi|^k\hat{u}\|^2_{L^2(\mathbb{R}^n)} \le C\sum_{\alpha=k}\|D^\alpha u\|^2_{L^2(\mathbb{R}^n)}$$

Thus an equivalent definition of $W^{k,2}$ is

$$H^k = \{u \in L^2 \big| \|u\|_{H^k} = \sum_{0\le l\le k} \||\xi|^k \hat{u}\|_{L^2(\mathbb{R}^n)} < \infty\}$$

We can also define the homogeneous (semi-) norms

$$\|u\|_{\dot{H}^k} = \||\xi|^k \hat{u}\|_{L^2(\mathbb{R}^n)}$$

There are semi-norms because $\|u\|_{\dot{H}^k} = 0$ does not imply $u = 0$. Polynomials of degree $\le k-1$ have $\dot{H}^k$ norm 0. We can get around this problem by requiring $u \in L^2$ or modding out polynomials.

Let us define H^s for $s \ge 0$, not necessarily an integer:

Definition 6.1. H^s is defined as

$$\{u \in L^2 \big| \|u\|_{H^s} = \|u\|_{L^2(\mathbb{R}^n)} + \||\xi|^s \hat{u}\|_{L^2(\mathbb{R}^n)} < \infty\}$$

Theorem 6.2. *Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The following sharp estimate holds for $s > 0$:*

$$\|u(x', 0)\|_{H^s(\mathbb{R}^{n-1})} \le C\|u\|_{H^{1/2+s}(\mathbb{R}^n)}$$

where $u \in \mathcal{S}$, but C is independent of u, thus the trace operator extends as a bounded linear operator from $H^1(\mathbb{R}^n)$ to $H^{1/2}(\mathbb{R}^{n-1})$.

Remark 6.3. This can be modified to show T is bounded from $H^1(U)$ to $H^{1/2}(\partial U)$ for C^1 bounded domains.

Proof. Let $\hat{u}(\xi', 0)$ denote the Fourier transform in the first $(n-1)$ variables, with $x_n = 0$ kept fixed. and let $\tilde{u}(\xi)$ denote the Fourier transform in all variables. We have

$$\hat{u}(\xi', 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(\xi) d\xi_n$$

thus we have the pointwise estimate (Cauchy-Schwarz)

$$\big||\xi'|^s \hat{u}(\xi', 0)\big|^2 \le \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \frac{|\xi'|^{2s}}{|\xi'|^{1+2s} + |\xi_n|^{1+2s}} d\xi_n \int_{-\infty}^{\infty} \left(|\xi'|^{1+2s} + |\xi_n|^{1+2s}\right) |\tilde{u}(\xi)|^2 d\xi_n$$

Check, using a change of variables, that the first integral is some finite C_s. Now integrate $d\xi'$:

$$\int_{\mathbb{R}^{n-1}} \big||\xi'|^s \hat{u}(\xi', 0)\big|^2 d\xi' \le \left(\frac{1}{2\pi}\right)^2 C_s \int_{\mathbb{R}^n} \left(|\xi'|^{1+2s} + |\xi_n|^{1+2s}\right) |\tilde{u}(\xi)|^2 d\xi \le C\|u\|_{H^{1/2+s}}^2$$

□

7. The Hardy-Littlewood-Sobolev estimates

Theorem 7.1. *Let $1 < \alpha < \infty$ and define*

$$Tf(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{\frac{n}{\alpha}}} f(y)dy$$

Let $1 < p < q < \infty$ satisfying

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{\alpha} - 1$$

Then $Tf(x)$ is finite for a.e x if $f \in L^p$ and there exists C depending only on n, p, q, α such that

$$\|Tf\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

Before proving this, notice that this says that $\frac{1}{|x-y|^{\frac{n}{\alpha}}}$ behaves as if it were in L^α from the point of view of Hausdorff-Young. Also, check that $\alpha = \frac{n}{n-2}$ and $\alpha = \frac{n}{n-1}$ generalize the Sobolev embedding theorem for $W^{2,p}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$.

Proof. I will follow Hedberg's short proof which depends on the Hardy-Littewood maximal function. Recall

$$Mf(x) = \sup_{r>0} \frac{1}{|B(r)|} \int_{B(x,r)} |f(y)|dy$$

and that there exists C such that

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p \leq \infty$. The proof has 3 steps.

Pick $\delta > 0$ and write

$$|Tf(x)| \leq \int_{|x-y|<\delta} \frac{1}{|x-y|^{\frac{n}{\alpha}}}|f(y)|dy + \int_{|x-y|>\delta} \frac{1}{|x-y|^{\frac{n}{\alpha}}}|f(y)|dy$$
$$= T_1(x) + T_2(x)$$

Step 1. Write

$$|T_1f(x)| = \sum_{i=0}^{\infty} \int_{\frac{\delta}{2^{i+1}}<|x-y|<\frac{\delta}{2^i}} \frac{1}{|x-y|^{\frac{n}{\alpha}}}|f(y)|dy$$
$$\leq C\delta^{n-\frac{n}{\alpha}} Mf(x) = C\delta^{\frac{n}{p}-\frac{n}{q}} Mf(x)$$

This follows from the definition of Mf and summing a geometric series.

Step 2.

$$|T_2f(x)| \leq C\delta^{-\frac{n}{\alpha}+\frac{n}{p'}}\|f\|_{L^p(\mathbb{R}^n)} = C\delta^{-\frac{n}{q}}\|f\|_{L^p(\mathbb{R}^n)}$$

This follows from Hölder's inequality and the conditions on p, q.

Step 3. For each x, optimize the choice of δ by making the two upper bounds equal:

$$\delta = \left(\frac{\|f\|_{L^p}}{Mf(x)}\right)^{\frac{p}{n}}$$

so that

$$|Tf(x)| \leq C(Mf(x))^{\frac{p}{q}} \|f\|_{L^p}^{1-\frac{p}{q}}$$

This implies the result. □

8. The fundamental solution of the Schrödinger equation

We already know how to solve the heat equation

$$(\frac{\partial}{\partial t} - \Delta)u(t,x) = 0 \text{ if } t > 0$$

with initial conditions $u(0,x) = f$. It is obtained by convolving in x with

$$\frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} H(t)$$

The Schrödinger equation is

$$(\frac{1}{i}\frac{\partial}{\partial t} - \Delta)u(t,x) = 0$$

and it is solved by convolving in x with

$$\frac{1}{(4\pi i t)^{n/2}} e^{\frac{-|x|^2}{4it}} H(t)$$

We have to decide which square root to use in n is odd. Also, we must prove

$$\lim_{t\to 0} \frac{1}{(4\pi i t)^{n/2}} \int e^{\frac{-|x-y|^2}{4it}} f(y)dy = f(x)$$

for sufficiently nice f. The reason behind this is stationary phase: let $T = \frac{1}{2t}$. In its simplest form, it states

$$\lim_{T\to\infty} T^{\frac{n}{2}} \int e^{iT\frac{|x|^2}{2}} f(x)dx = (2\pi i)^{n/2} f(0)$$

How much regularity we need for f and which square root of i is used will drop out of the proof (next time). We need some facts about the Fourier transform.

9. The method of stationary phase

Theorem 9.1. *Let $f \in \mathcal{S}$. Then*

$$\lim_{T\to\infty} T^{\frac{n}{2}} \int e^{iT\frac{|x|^2}{2}} f(x)dx = (2\pi i)^{n/2} f(0)$$

where $i^{n/2} = e^{\frac{in\pi}{4}}$.

Proof.

$$\int e^{iT|x|^2} f(x)dx = \lim_{\epsilon\to 0} \int e^{(-\epsilon+iT)\frac{|x|^2}{2}} f(x)$$
$$=(2\pi)^{-n} \lim_{\epsilon\to 0} \int e^{(-\epsilon+iT)\frac{|x|^2}{2}} \hat{\hat{f}}_-(x)$$

Since $f(0) = f_-(0)$, we don't have to carry the minus sign.

$$A_\epsilon = \int e^{(-\epsilon+iT)\frac{|x|^2}{2}} \hat{\hat{f}}(x)dx$$
$$= \int \mathcal{F}\left(e^{(-\epsilon+iT)\frac{|x|^2}{2}}\right)(\xi)\hat{f}(\xi)d\xi$$

Now we do the analytic continuation with which we are familiar by now:

$$\mathcal{F}\left(e^{-\frac{|x|^2}{2}}\right) = (2\pi)^{n/2} e^{-\frac{|\xi|^2}{2}}$$
$$\mathcal{F}\left(e^{-\frac{a|x|^2}{2}}\right) = (2\pi)^{n/2} \frac{1}{a^{n/2}} e^{-\frac{|\xi|^2}{2a}}$$

for $a > 0$ and therefore for $\Re a > 0$, with the usual branch of the argument function. Take $a = \epsilon - iT$. We get

$$\mathcal{F}\left(e^{(-\epsilon+iT)\frac{|x|^2}{2}}\right) = (2\pi)^{n/2} \frac{1}{(\epsilon - iT)^{n/2}} e^{\frac{-|\xi|^2}{2(\epsilon-iT)}}$$

So

$$A_\epsilon = (2\pi)^{n/2} \frac{1}{(\epsilon - iT)^{n/2}} \int e^{\frac{-|\xi|^2}{2(\epsilon-iT)}} \hat{f}(\xi)d\xi$$
$$\to A = (2\pi T^{-1})^{n/2} e^{i\frac{n\pi}{4}} \int e^{\frac{|\xi|^2}{2iT}} \hat{f}(\xi)d\xi$$

and the quantity in the statement of the theorem,

$T^{\frac{n}{2}} \int e^{iT\frac{|x|^2}{2}} f(x)dx$ equals

$T^{n/2}(2\pi)^{-n}A$

$= (2\pi)^{-n/2} e^{i\frac{n\pi}{4}} \int e^{\frac{|\xi|^2}{2iT}} \hat{f}(\xi) d\xi$

$= \left((2\pi)^{n/2} e^{i\frac{n\pi}{4}}\right) \frac{1}{(2\pi)^n} \int \left(1 + O(\frac{|\xi|^2}{T})\right) \hat{f}(\xi) d\xi \to \left((2\pi)^{n/2} e^{i\frac{n\pi}{4}}\right) f(0)$

by the dominated convergence theorem, provided $\int (1+|\xi|^2)|\hat{f}(\xi)|d\xi \leq C$. This tells us how much regularity we need for this proof to work. We used $|e^{ix} - 1| \leq C|x|$). □

There is an obvious generalization by Taylor expanding $e^{\frac{|\xi|^2}{2iT}}$.

10. Estimates for the Schrödinger equation

In this section we look at solutions to Schrödinger equation

$$\begin{aligned}&(\frac{1}{i}\frac{\partial}{\partial t} - \Delta)u(t,x) = 0\\ &u(0,x) = f_0(x)\end{aligned}$$

where, for the time being, $f_0 \in \mathcal{S}(\mathbb{R}^n)$. The solution can be written (formally, if you wish, or using the Fourier transform) as $u = e^{it\Delta} f_0$.

One way of writing down the solution (which is unique) is

$$u(t,x) = \frac{e^{-n\pi i/4}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4it}} f_0(y)dy$$

which gives

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}} \|f_0\|_{L^1}$$

Another way of representing the solution is

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{f}_0(\xi)$$

From here we get

$$\|u(t,\cdot)\|_{L^2} = \|f_0\|_{L^2}$$

Recall the Riesz-Thorin complex interpolation theorem:

Theorem 10.1. *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$. Let T be a linear operator (initially defined on a dense subset) such that*

$$\|T(f)\|_{L^{q_1}} \le C_1 \|f\|_{L^{p_1}}$$
$$\|T(f)\|_{L^{q_2}} \le C_1 \|f\|_{L^{p_2}}$$

Let $0 \le \theta \le 1$, and define

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$
$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$
$$C = C_1^{\theta} C_2^{1-\theta}$$

Then

$$\|T(f)\|_{L^q} \le C \|f\|_{L^p}$$

for all $f \in L^p$.

Applying this to $T(f_0) = u(t,\cdot)$ we get, for all $2 \le q \le \infty$,

$$\|u(t,\cdot)\|_{L^q} \le \frac{1}{(4\pi t)^{\frac{n(q-2)}{2q}}} \|f_0\|_{L^{q'}} \tag{3}$$

This leads to an example of a Strichartz estimate.

Theorem 10.2. *Let*

$$q = \frac{2(n+2)}{n}$$

Then

$$\|u\|_{L^q(\mathbb{R}^{n+1})} \le C \|f_0\|_{L^2(\mathbb{R}^n)}$$

Proof. By duality, it suffices to show

$$\left| \int \left(e^{it\Delta} f_0\right)(x) G(t,x)\, dt dx \right| \le C \|f_0\|_{L^2(\mathbb{R}^n)} \|G\|_{L^{q'}(\mathbb{R}^{n+1})}$$

This is equivalent to showing

$$A = \left| \int f_0(x) \left(e^{it\Delta} G(t,\cdot)\right)(x)\, dt dx \right| \le C \|f_0\|_{L^2(\mathbb{R}^n)} \|G\|_{L^{q'}(\mathbb{R}^{n+1})}$$

which, in turn, is equivalent to

$$B = \left\| \int e^{it\Delta} G(t,\cdot)\, dt \right\|_{L^2(\mathbb{R}^n)} \le C \|G\|_{L^{q'}(\mathbb{R}^{n+1})} \tag{4}$$

Doubling, this is the same as

$$B^2 = \left| \int \left(\int e^{it\Delta} G(t,\cdot)\, dt \right) \overline{\left(\int e^{is\Delta} G(s,\cdot)\, ds \right)} dx \right| \le C^2 \|G\|^2_{L^{q'}(\mathbb{R}^{n+1})}$$

The LHS is

$$B^2 = \left| \int\int\int \left(e^{i(t-s)\Delta}G(t,\cdot)\right)\overline{G(s,x)}dtdsdx\right|$$

By Fubini and Hölder's inequality, B^2 is dominated by

$$D = \int\int \|e^{i(t-s)\Delta}G(t,\cdot)\|_{L^q(\mathbb{R}^n)}\|G(s,\cdot)\|_{L^{q'}(\mathbb{R}^n)}dtds$$

Now apply (3) with $q = \frac{2(n+2)}{n}$ so that $\frac{n(q-2)}{2q} = \frac{n}{n+2}$.

$$\|e^{i(t-s)\Delta}G(t,\cdot)\|_{L^q(\mathbb{R}^n)} \le \frac{C}{|t-s|^{\frac{n}{n+2}}}\|G(t,\cdot)\|_{L^{q'}(\mathbb{R}^n)}$$

so that

$$D \le \int\int \frac{C}{|t-s|^{\frac{n}{n+2}}}\|G(t,\cdot)\|_{L^{q'}(\mathbb{R}^n)}\|G(s,\cdot)\|_{L^{q'}(\mathbb{R}^n)}dtds$$

Now, $q' = \frac{2(n+2)}{n+4}$ and $\frac{1}{q'} + \frac{n}{n+2} - 1 = \frac{1}{q}$ so, by HLS

$$\| \int \frac{1}{|t-s|^{\frac{n}{n+2}}}\|G(s,\cdot)\|_{L^{q'}(\mathbb{R}^n)}ds\|_{L^q(dt)} \le C\|G(\cdot,\cdot)\|_{L^{q'}(\mathbb{R}^{n+1})} \tag{5}$$

so, finally,

$$D \le \|G(\cdot,\cdot)\|^2_{L^{q'}(\mathbb{R}^{n+1})}$$

□

We have actually also found an estimate for the inhomogeneous equation

$$\begin{aligned}&(\frac{1}{i}\frac{\partial}{\partial t} - \Delta)u(t,x) = F(t,x)\ (t>0, x\in\mathbb{R}^n)\\ &u(0,x) = 0\end{aligned}$$

The solution is given by Duhamel's formula

$$u(t,x) = i\int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds$$

and we have

Proposition 10.3. *Let* $q = \frac{2(n+2)}{n}$. *There exists* C *such that*

$$\|u\|_{L^q(dtdx)} \le C\|F\|_{L^{q'}(dtdx)}$$

Proof.

$$\begin{aligned}\|u\|_{L^q(dtdx)} &= \Big\| \int_0^t e^{i(t-s)\Delta} F(s,\cdot)ds \Big\|_{L^q(dtdx)} \\ &\le \Big\| \int_0^t \|e^{i(t-s)\Delta}F(s,\cdot)\|_{L^q(dx)} ds \Big\|_{L^q(dt)} \\ &\le C \Big\| \int_0^\infty \frac{1}{|t-s|^{\frac{n}{2}(1-\frac{2}{q})}} \|F(s,\cdot)\|_{L^{q'}(dx)} ds \Big\|_{L^q(dt)} \\ &\le C\|F\|_{L^{q'}(dtdx)}\end{aligned}$$

by HLS, just as in (5). □

Remark 10.4. The easiest explanation for the exponent $q = \frac{2(n+2)}{n}$ is scaling. There is a much more subtle reason (the "Knapp counterexample"). On the Fourier transform side, (4) reads

$$\|\hat{G}(|\xi|^2, \xi)\|_{L^2(d\xi)} \le C\|G\|_{L^{q'}(\mathbb{R}^{n+1})} \tag{6}$$

This is an instance of the Stein-Tomas restriction theorem. It is clear that the Fourier transform of any L^1 function has a well-defined restriction to a hypersurface (because it is continuous). It is also clear that the Fourier transform of general L^2 functions don't have well-defined restrictions to a hypersurface (because they are just L^2). We have just proved that any $L^{q'}$ function has a well-defined restriction to the hyperboloid. The Knapp counterexample shows that if (6) holds for all G such that $\hat{G}$ is supported in the unit ball, then $q' \le \frac{2(n+2)}{n+4}$. Let ϕ be a fixed even bump function, $\delta > 0$ small, and take

$$\hat{G}(\tau, \xi) = \phi(\frac{\tau}{\delta^2})\phi(\frac{\xi_1}{\delta}) \cdots \phi(\frac{\xi_n}{\delta})$$

then

$$G(t,x) = \frac{1}{(2\pi)^{n+1}} \delta^{n+2} \hat{\phi}(\delta^2 t)\hat{\phi}(\delta x_1) \cdots \hat{\phi}(\delta x_n)$$

With this choice, the LHS of (6) is $\sim \delta^{n/2}$ (draw a picture), while the RHS is $\sim \delta^{\frac{n+2}{q}}$ so $q \ge \frac{2(n+2)}{n}$.