Math 130 – Spring 2015 – Boyle –Exam 1 – Solutions

1. (13 points)

(a) (4 pts) What is the domain of $y = \ln(x-6)$? What is the range?

SOLUTION.

The domain is $(6, \infty)$; i.e., all real numbers greater than 6.

The range (set of outputs) is $(-\infty, \infty)$, i.e., all real numbers.

(b) (3 pts) Solve $\log_2(x) = -3$.

SOLUTION.

$$\begin{aligned} \log_2(x) &= -3 \\ 2^{\log_2(x)} &= 2^{-3} \\ x &= 1/8 \end{aligned}$$

(c) (6 pts) Solve $\sqrt{e^x} = e^x e^{x+1}$.

SOLUTION.

$$\begin{aligned} \sqrt{e^x} &= e^x e^{x+1} \\ e^{x/2} &= e^x e^{x+1} \\ e^{x/2} &= e^{2x+1} \\ x/2 &= 2x+1 \\ -1 &= \frac{3}{2}x \\ x &= -\frac{2}{3} \, . \end{aligned}$$

2. (10 points)

Potassium-40, with a half-life of 1.25 billion years, has been used by geochronologists trying to sort out the mass extinction of 250 million years ago. What fraction of the Potassium-40 remains from a creature that died 250 million years ago?

SOLUTION.

Let t have units of billions of years. Let $t = 0$ be the time 250 million (.25 billion) years ago. Let $y(t)$ be the amount of potassium-40 in the creature at time t. The fraction we need to compute is $y(.25)/y(0)$.

$$\begin{aligned}
y(t) &= y(0)e^{kt} \\
y(t) &= y(0)e^{(-\ln 2)(1/1.25)t} \\
y(.25) &= y(0)e^{(-\ln 2)(1/1.25)(.25)} \\
\frac{y(.25)}{y(0)} &= e^{(-\ln 2)(1/1.25)(.25)} \\
&= e^{(-\ln 2)(1/5)} \;=\; 2^{-1/5} \;=\; 1/\sqrt[5]{2}\ .
\end{aligned}$$

Alternately, you can just notice that 250 million years is one fifth of the half life, and therefore the answer is $2^{-1/5}$.

3. (10 points)

(a) (4 pts) Find all values of x between 0 and 2π for which $\sin x = 1/2$.

SOLUTION: $x = \pi/6$ and $x = 5\pi/6$

(b) (3 pts) What is the period of the function $y = 5\sin(3t + 2)$?

SOLUTION: $2\pi/3$

(c) (3 pts) For the population model function

$$P(t) = \frac{8}{1 + 4e^{-3t}}$$

compute $\lim_{t\to\infty} P(t)$.

SOLUTION: 8

4. (12 points) Let position be measured in feet and let time be measured in seconds. Suppose the position of an object moving in a straight line is given by $s(t) = |t|$.

(a) (6 pts) What is the average velocity between $t = -3$ and $t = 1$?
SOLUTION.

$$\frac{s(1) - s(-3)}{1 - (-3)} = \frac{1 - 3}{1 + 3} = -\frac{1}{2} \text{ ft/sec} .$$

(b) (3 pts) What is the instantaneous velocity at $t = 2$?
SOLUTION.
1 ft/sec.
(For $t > 0$, $|t| = t$, and therefore $s'(t) = 1$ if $t > 0$.)

(c) (3 pts) What is the instantaneous velocity at $t = -2$?
SOLUTION.
-1 ft/sec.
(For $t < 0$, $|t| = -t$, and therefore $s'(t) = -1$ if $t < 0$.)

5. (16 points) Determine the following limits. (4 points each)

$(a)\ \lim_{x\to 0} f(x)$ with f defined by $f(x) = \begin{cases} 3x+1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

$(b)\ \lim_{x\to -1} \dfrac{x^2+4x+3}{x+1}$

$(c)\ \lim_{x\to \frac{\pi}{2}^-} \tan(x)$

$(d)\ \lim_{x\to -\infty} \dfrac{3x^5+5x^4-19,000}{5x^5+4x^4+3x^3}$

SOLUTION.

$$(a)\ \lim_{x\to 0} f(x) = \lim_{x\to 0} 3(0)+1 = 1$$

$$(b)\ \lim_{x\to -1} \frac{x^2+4x+3}{x+1} = \lim_{x\to -1} \frac{x^2+4x+3}{x+1} = \lim_{x\to -1} (x+3) = 2$$

$$(c)\ \lim_{x\to \frac{\pi}{2}^-} \tan(x) = \infty$$

$$(d)\ \lim_{x\to -\infty} \frac{3x^5+5x^4-19,000}{5x^5+4x^4+3x^3} = \lim_{x\to -\infty} \frac{3x^5}{5x^5} = \frac{3}{5}\ .$$

6. (12 points) (a) (8 pts) A tumor is approximately spherical in shape. If the radius of the tumor grows from 13 mm to 15 mm, what is the linear approximation to the change in the volume of the tumor?

SOLUTION.
Let $V = V(r)$ be the volume of a ball of radius R mm. We want the linear approximation to $V(15) - V(13)$. We have $V(r) = (4/3)\pi r^3$ and therefore $V'(r) = 4\pi r^2$. So ...

$$\begin{aligned} V(13) - V(15) &\approx V'(13)(15 - 13) \\ V(13) - V(15) &\approx 4\pi(13)^2(2) = 8(169)\pi \\ V(13) - V(15) &\approx (1352)\pi \text{ mm}^3 \;. \end{aligned}$$

(b) (4 pts) Given an example of a continuous function $y = f(x)$, from $(-\infty, \infty)$ to $(-\infty, \infty)$, and a number a such that $f'(a)$ does not exist. You do not have to give a proof.

SOLUTION.
$f(x) = |x|$ and $a = 0$.
(There are infinitely many other solutions.)

7. (17 points) (a) (9 pts) Find an equation for the tangent line of the graph of $y = \sqrt{x}$ at the point $(9, 3)$.

SOLUTION.
$y = x^{1/2}$, so $y' = (1/2)x^{-1/2} = 1/(2\sqrt{x})$. We can get an equation for the tangent line by substituting into the point-slope form:

$$\begin{aligned} y - y_0 &= (\text{slope})(x - x_0) \\ y - 3 &= \frac{1}{2\sqrt{9}}(x - 9) \\ y - 3 &= \frac{1}{6}(x - 9) \ . \end{aligned}$$

Determine the following (2 pts each).

$$\begin{aligned} &(b) \lim_{x \to +\infty} \frac{\sin(x)}{x} &&= 0 \\ &(d) \lim_{x \to +\infty} \frac{x^5}{e^x} &&= 0 \\ &(c) \lim_{x \to +\infty} \sin(x) && DNE \\ &(e) \lim_{x \to +\infty} \frac{\sqrt{x}}{\ln(x)} &&= \infty \end{aligned}$$