Math 130 – Spring 2015 – Boyle –Exam 2–Solutions

• NO CALCULATORS OR ELECTRONIC DEVICES ALLOWED.
• Where a calculator would be used, give your answer as an expression a calculator could evaluate. For full credit, simplify expressions appropriately.
• Use a separate answer sheet for each of the SEVEN questions.
• Give your pledge on page 1 only, covering the whole test.
• Draw a box around a final answer to a problem.

1. (14 points)
For each of the following functions, find the formula for y'.
(a) (7 pts) $y = 2^{-5x}$.
Solution.

$$y = 2^{-5x} = e^{(\ln 2)(-5x)}$$
$$y' = (\ln 2)(-5)e^{(\ln 2)(-5x)} = -(5\ln 2)2^{-5x} \ .$$

(b) (7 pts) $y = \ln(|\sin(2x)|)$.
Solution.

$$y = \ln(|\sin(2x)|)$$
$$y' = \frac{2\cos(2x)}{\sin(2x)} = 2\cot(2x) \ .$$

2. (14 points)
(a) (7 pts) Given $y = \log_{10}(\sqrt{3x})$, find the formula for y'.
Solution.

$$y = \log_{10}(\sqrt{3x}) = \log_{10}(3x)^{1/2} = \frac{1}{2}\log_{10}(3x) = \frac{1}{2\ln(10)}\ln(3x)$$
$$y' = \frac{1}{2\ln(10)}\frac{3}{3x} = \frac{1}{2x\ln(10)} \ .$$

(b) (7 pts) Given $y = (\cos(x))/(x^2+1)$, find the formula for y'.
Solution.

$$y' = \frac{(x^2+1)(-\sin x) - (\cos x)(2x)}{(x^2+1)^2} \ .$$

3. (14 points)
Find every relative extreme value of the function $f(x) = (\ln x)(x^2)$, and indicate which are relative maxima and which are relative minima. (Remember, values are outputs.)
Solution.

$$\begin{aligned} f'(x) &= (\ln x)'(x^2) + (\ln x)(x^2)' \\ &= (1/x)(x^2) + (\ln x)(2x) \\ &= x + (\ln x)(2x) = x(1 + 2\ln x) \ . \end{aligned}$$

The domain of f is $(0, \infty)$. $f'(x)$ is zero only at $x = e^{-1/2} = 1/\sqrt{e}$.
If $0 < x < 1/\sqrt{e}$, then $f'(x) < 0$; of $1/\sqrt{e} < x < \infty$, then $f'(x) > 0$.
By the first derivative test, at $x = 1/e$ f has a relative minimum value, which is $f(1/\sqrt{e}) = (-1/2)(1/e) = -1/2e$. This is the only relative extreme value.

4. (14 points) For each of the following functions, determine all asymptotes; if there is no asymptote for a function, say so.

$$f(x) = 7x + \frac{\cos x}{x} \qquad g(x) = \frac{8x^4 + 3x + 1}{x^2 + 5} \qquad h(x) = \ln(x) \ .$$

Solution.

- f has an oblique asymptote $y = 7x$.
- f has a vertical asymptote at $x = 0$.
- g has no asymptote.
- h has a vertical asymptote at $x = 0$.

5. (14 points) (4 pts)
The formulas for the volume V and surface area A of a ball as a function of its radius R are $V = \frac{4}{3}\pi R^3$ and $A = 4\pi R^2$. There are numbers C and s such that $A = CV^s$ gives the area of a ball as a function of its volume.

a. (2 pts) What is the relationship between A and dV/dR?

b. (4 pts) What is s?

c. (4 pts) Find the formula which gives dA/dV as a function of V. (You do not have to solve for C, but you must use the correct number for s.)

d. (4 points) Compute $\lim_{R\to\infty} dA/dR$ and $\lim_{V\to\infty} dA/dV$.

Solutions.

a. $A = dV/dR$.

b. $s = 2/3$.
This problem was about a theme in one of the biology worksheets. Here $s = 2/3$ comes from $(R^3)^{2/3} = R^2$. In more detail:

$$V^{2/3} = (\frac{4}{3}\pi R^3)^{2/3} = (\frac{4}{3}\pi)^{2/3}(R^3)^{2/3} = (\frac{4}{3}\pi)^{2/3}(R^2) \ .$$

Then

$$A = 4\pi R^2 = \left(4\pi \frac{1}{(\frac{4}{3}\pi)^{2/3}}\right) V^{2/3} \ .$$

So, $s = 2/3$ and C is that messy number multiplying $V^{2/3}$.

c. Since $A = CV^{2/3}$, we have $dA/dV = (2/3)CV^{-1/3}$.

d.

$$\lim_{R\to\infty} dA/dR = \lim_{R\to\infty} (8\pi R) = \infty$$

$$\lim_{V\to\infty} dA/dV = \lim_{V\to\infty} (2/3)C\frac{1}{\sqrt[3]{V}} = 0 \ .$$

6. (14 points) For a given positive constant r, the Ricker model of population uses the function $P(x) = xe^{r(1-x)}$ to estimate the population one year from today, given that the population now (in suitable units) is x. The domain of P is $[0, \infty)$.
(a) (2 pts) Find all asymptotes for P (if there are none, say so).
(b) (4 pts) Find the intervals on which f is increasing/decreasing. (c) (2 pts) Determine all inputs x at which f has a relative maximum or minimum (say which).
(d) (2 pts) You may assume $P''(x) = (e^{r(1-x)})(-2r + r^2x)$. Find the intervals on which the graph of f is concave up/down.
(e) (4 points) For the parameter value $r = 1$, graph f.
Solutions. (a) $y = 0$ is a horizontal asymptote for P .
(b)

$$\begin{aligned} P'(x) &= (x)'(e^{r(1-x)}) + (x)(e^{r(1-x)})' \\ &= (e^{r(1-x)}) + (x)(-re^{r(1-x)}) \\ &= e^{r(1-x)}(1 - rx) \end{aligned}$$

From the sign of P': f is increasing on $[0, 1/r]$ and decreasing on $[1/r, \infty)$.

(c) f assumes a relative max at $x = 1/r$.
(d) By the way, here is a computation of $P''(x)$:

$$\begin{aligned} P''(x) &= (e^{r(1-x)})'(1 - rx) + (e^{r(1-x)})(1 - rx)' \\ &= (-re^{r(1-x)})(1 - rx) + (e^{r(1-x)})(-r) \\ &= (e^{r(1-x)})((-r + r^2x) + (-r) \\ &= (e^{r(1-x)})(-2r + r^2x) = (e^{r(1-x)})(r)(-2 + rx) \ . \end{aligned}$$

$f'', 0$ on $(0, 2/r)$ and $f'' > 0$ on $(2/r, \infty)$. So, the graph of f is concave down on $(0, 2/r)$ and concave up no $(2/r, \infty)$.

(e) The graph is not included for technical reasons. We can write P also as $P(x) = (e^r)xe^{-rx}$. So, the graph will be a rescaling of the graph of $y = xe^{-x}$.

7. (16 points)

(a) (4 pts) You are given the following table of values:

x	1	2	3	4
$f(x)$	2	4	1	3
$f'(x)$	-6	-7	-8	-9
$g(x)$	2	3	4	1
$g'(x)$	2/7	3/7	4/7	5/7

If $h(x) = g(f(x))$, what is $h'(1)$?

Solution.

(a) By the chain rule,
$h'(1) = [g'(f(1))][f'(1)] = [g'(2)][f'(1)] = [3/7][-6] = -18/7$.

(b) Answer each of the following TRUE or FALSE. No proof required.

(i) (4 pts) The largest number of local extreme values a polynomial of degree 5 can have is 5.

FALSE. If a polynomial f has an extreme value at x, then $f'(x) = 0$. Here f' is a degree four polynomial. It cannot have more than four roots (a non-constant polynomial of degree k has at most k distinct roots).

(ii) (4 pts) If f is the function $f(x) = e^x$, then $f'(2x) = 2f'(x)$, for every x.

FALSE.

(iii) (4 pts) In the Fitz-Hugh-Nagumo model of neuron communication, the rate of change of the electrical potential with respect to time is given as a function of the potential v by $f(v) = v(a - v)(v - 1)$. Suppose $a = 1/4$.

True or False: The electrical potential is increasing with respect to time when $v = 1/5$.

FALSE. $f(1/5) = (1/5)(1/4-1/5)(1/5-1)$. The sign of $f(1/5)$ is $(+)(+)(-) = (-)$. Since $f(1/5) < 0$, the electrical potential is increasing with respect to time when $v = 1/5$.