Math 131 – Spring 2016 – Boyle –Exam 1

- NO CALCULATORS OR ELECTRONIC DEVICES ALLOWED.
- Use a separate answer sheet for each question; use the back side of an answer sheet if you need more space to answer a question.
- Give your pledge on page 1 only, covering the whole test.
- Draw a box around a final answer to a problem.

1. (10 points) For k a positive constant, compute the improper integral $\int_{x=0}^{\infty} e^{-kx}\,dx$ (which you've seen approximating the amount of carbon fixed in photosynthesis below a given area of ocean).

2. (14 points) Compute $\int_{x=0}^{\pi} x \sin 3x\,dx$.

3. (12 points) (a) (8 pts) Compute $\int_{x=1}^{\infty} (1/\sqrt[3]{x})\,dx$.
(A correct answer for (a) is either a real number or $+\infty$.)
(b) (4 pts) For which numbers p is $\int_{x=1}^{\infty} \frac{1}{x^p}\,dx < \infty$? (No proof required.)

4. (12 points) Suppose for a population model that $N(t)$, the population N at time t, is assumed to satisfy the differential equation

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N$$

where r and K are positive constants. (N must be nonnegative.)
(a) (2 pts) Determine the equilibrium values of N.
(b) (6 pts) Draw the phase diagram.
(c) (4 pts) Determine whether the equilibria are stable or unstable.

5. (12 points) Suppose a glucose solution is administered intravenously at a constant rate r. To model the glucose concentration $C(t)$ at time $t \geq 0$, suppose

$$\frac{dC}{dt} = -k(C - r)$$

where r and k are positive constants.
(a) (9 pts) Solve the differential equation to find a formula for $C(t)$ in terms of k, r and C_0 (where C_0 denotes the concentration at time $t = 0$).
(b) (3 pts) What is $\lim_{t\to\infty} C(t)$?

6. (14 points) Find the solution of the differential equation that statisfies the given initial condition:

$$\frac{dy}{dx} = \frac{x}{y}\ , \quad y(0) = -5\ .$$

7. (12 points) The system of differential equations

$$\frac{dx_1}{dt} = 2x_1 - 3x_1x_2 \qquad \text{and} \qquad \frac{dx_2}{dt} = -3x_2 + 2x_1x_2$$

is chosen such that $x_1(t)$ and $x_2(t)$ model the sizes of two populations as a function of time. (Population size can be zero or positive.)
(a) (4 pts) What are the equilibrium points of the system?
(b) (4 pts) Assume for every vertical line L that no solution curve can hit the line more than twice. Sketch some solution curves in the first quadrant, with arrows indicating the direction of motion.
(c) (4 pts) Determine whether each equilibrium point is stable or unstable.

8. (14 points) Suppose two spheres are given by the following equations: $(x-9)^2 + (y-3)^2 + (z-1)^2 = 4$ and $(x-6)^2 + (y-6)^2 + (z+2)^2 = 9$.
(a) (8 pts) Compute the centers and radii of these spheres.
(b) (6 pts) Do these spheres overlap? (This kind of question arose when we considered antigenic evolution and vaccination.) Show enough work that we can follow your logic.