MATH 240 – Spring 2013 – Final Exam

CALCULATORS ARE <u>NOT</u> ALLOWED
TURN OFF ALL ELECTRONIC DEVICES

There are 5 questions. Answer each on a separate sheet of paper. Use the back side if necessary.
On each sheet, put your name and your section TA and meeting time.
You may assume given matrix equations are well defined (i.e. the matrix sizes are compatible).
If your final answer is short, put a BOX around it.

1. (a) (25 points) The following matrices are row equivalent:

$$W = \begin{pmatrix} 4 & 2 & -1 & 4 & 7 \\ 3 & 1 & 0 & 2 & 3 \\ 1 & 1 & -1 & 2 & 4 \\ 2 & 2 & -3 & 0 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 8 & 18 \\ 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) [7 pts] Write down a basis for the row space of W.
   (b) [7 pts] Write down a basis for the column space of W.
   (c) [8 pts] Write down a basis for the null space of W.
   (d) [3 pts] What is the rank of W?

   (b) (15 points) TRUE or FALSE? No justification required.

   i. The set of vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ such that $x_1x_2x_3 = 0$ is a subspace of $\mathbb{R}^3$.
   ii. The set of polynomials $\{p_1, p_2, p_3, p_4\} = \{2+t,\ 3t+t^2,\ -7+t+4t^2,\ -1+9t+8t^2\}$ is linearly independent.
   iii. Let V be the vector space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and define the function T from V to $\mathbb{R}$ by the rule $T(f) = f(2)$. Then T is a linear transformation.

2. (a) (10 points) Suppose A, B, C are invertible 3×3 matrices and as a partitioned matrix (block matrix) the 6×6 matrix U has the form $U = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$.

   Find the block matrix form for U^{-1}.

   (b) (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates a vector counterclockwise around the origin by $5\pi/4$ radians and then multiplies its length by 3. Find the matrix A such that $T(x) = Ax$ for all x.

   (c) (15 points) Below, U is an $n \times n$ orthogonal matrix.

   i. Define what it means for U to be an orthogonal matrix.
   ii. Suppose z is a vector in $\mathbb{R}^n$. Prove that $||Uz|| = ||z||$.
   iii. Suppose x and y are vectors in $\mathbb{R}^n$. Prove that $\text{dist}(Ux, Uy) = \text{dist}(x, y)$ (where e.g. $\text{dist}(x, y)$ denotes the distance from x to y).

***** THERE ARE QUESTIONS ON BOTH SIDES OF THIS PAPER *****

3. Define

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix} .$$

(a) (25 points) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, or prove that no such matrices exist.

(b) (15 points) TRUE or FALSE? No justification required.

i. If A is a matrix, then $A^T A$ is a symmetric matrix.

ii. For a matrix A and an invertible matrix U, $\text{rank}(UA) = \text{rank}(A)$.

iii. If $A = [2\ 2\ ;\ 2\ 2]$ (in MATLAB notation) and $S = \{Ax : ||x|| \leq 1\}$, then the area of S equals 4π.

4. (a) (10 points) For an $m \times n$ matrix A, write the equation relating the rank of A and the dimension of its null space.

(b) (10 points) Suppose that $(t-2)^2(t-1)(t+2)t$ is the characteristic polynomial of a matrix M, and the null space of $M - 2I$ has dimension 2.
Which of (a), (b) or (c) below is the correct statement? (No justification required.)
(a) M must be diagonalizable.
(b) M might be diagonalizable.
(c) M cannot be diagonalizable.

(c) (25 points) Find the numbers β_0, β_1 such that the line $x_2 = \beta_0 + \beta_1 x_1$ is the least squares best fit to the data points $(0,4), (1,3), (3,2), (4,1)$.

5. (a) (10 points) TRUE or FALSE?

i. If A is a 2×2 matrix with rank 2, then $\{Ax : ||x|| = 1\}$ is an ellipse.

ii. The function $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 3x + 1$ is a linear transformation.

(b) (5 points) What is the 3×3 elementary matrix E such that EA is obtained by adding 5 times row 3 of A to row 1 of A?

(c) (10 points) Give the definition of the dimension of a finite dimensional vector space.

(d) (15 points) For the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, what is the maximum length of a vector Bx, under the constraint that $||x|| = 1$?