MATH 405 – Upcoming Exam 3, Friday May 7

This exam will cover what we did after Exam 2 through last Friday (material covered from Chapters 7-11). It's a lot. The final is cumulative, and some of this will be there.

Overall this exam will be like Exam 2, rather long, but (I hope) with even more high scores. Below is a brief outline of material and some comments on questions.

Chapters 7,8.

Of course you must know what similarity of matrices means; what the characteristic polynomial is and how it is related to eigenvalues and similarity.

Scalar product, positive definite scalar product, transpose operator, symmetric operator real unitary matrix (i.e., orthogonal matrix).

Hermitian product (or form), positive definite Hermitian product, adjoint operator A*, Hermitian operator, complex unitary matrix (i.e., unitary matrix).

Quadratic form.

Hermitian operator A*=A Hermitian form, matrix

Diagonalizability.

A matrix A is diagonalizable if there is an invertible matrix U such that $U^{-1}AU$ is a diagonal matrix D. In this case, column j of U must be an eigenvector for eigenvalue the jth diagonal entry of D (as seen by considering column j on both sides of the equation $AU = UD$). Diagonalizability depends on the field $\mathbb{K}$ under consideration.

A square matrix A over $\mathbb{C}$ is diagonalizable if and only if for every eigenvalue λ, the multiplicity of λ as a root of the characteristic polynomial equals the dimension of its eigenspace, Kernel$(A-\lambda I)$. ("The algebraic multiplicity of λ equals its geometric multiplicity."). This is a consequence of similarity to a matrix in Jordan form.

If A is a symmetric matrix over $\mathbb{R}$, then there is a real unitary matrix U such that $U^{-1}AU$ is diagonal. (In particular, eigenvalues of A are real.)

If A is a Hermitian matrix ($A = A^*$) over $\mathbb{C}$, then there is a complex unitary matrix U such that $U^{-1}AU$ is diagonal. The eigenvalues of A again must be real.

If A is a complex unitary matrix A, there is a real unitary U such that $U^{-1}AU$ is a diagonal matrix. The complex numbers which are eigenvalues of complex unitary matrices are the numbers on the unit circle (example: $A = [e^{i\theta}]$).

If A is a real unitary matrix A, there is a real unitary U such that $U^{-1}AU$ is a block diagonal matrix, where the possible diagonal blocks are those of the form $[1], [-1]$ and M_θ, where M_θ is the rotation matrix $\begin{pmatrix} \cos(\theta) & -\sin(\theta \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

There is a real unitary matrix U such that $U^{-1}AU$ is diagonal if and only if A is symmetric.

There is a complex unitary matrix U such that $U^{-1}AU$ is diagonal if and only if A is normal (that is, $AA^* = A^*A$, where A^* is the conjugate transpose). Normal matrices can have any eigenvalues (if A is a 1×1 matrix, then $AA^* = A^*A$).

(Terminology. "Spectral Theorem" referred to a result stating that for a class of matrices A, there is a real or complex unitary matrix U such that $U^{-1}AU$ is diagonal.)

Ch.9 polynomials.

This was basically about understanding $p(A)$, when A is a square matrix and $p(t)$ is a polynomial. In particular, for any matrix B (including $B = tI - A$), we ahve $p(U{-}1BU) = U^{-1}p(B)U$, and this was used to show $(U{-}1AU)$ and A have the same characteristic polynomial.

Ch. 10

If a matrix A has all roots in $\mathbb{K}$, then it is similar to an upper triangular matrix. Equivalently, it has a fan basis. If $\mathbb{K} = \mathbb{C}$, then U can be chosen complex unitary. A is similar to a triangular matrix if and only if it has a fan basis: a basis $v_1, \ldots, v_n$ such that for each j, the j dimensional subspace spanned by $v_1, ..., v_j$ is mapped into itself by A.

Know the Cayley-Hamilton Theorem !

Ch. 11

Euclidean algorithm for polynomials. How to compute gcd(f,g) as polynomial combination. The proof that irreducible implies prime in $\mathbb{K}[t]$. Unique factorization in $\mathbb{K}[t]$. In $\mathbb{C}[t]$. Given the matrix A, the proof that $\mathbb{K}^n$ is a direct sum of the subspaces $V_i = \mathrm{Ker}(A - \lambda_i I)^{n_i}$, where n_i is the algebraic multiplicity of λ_i as a root of the characteristic polynomoial $\chi_A(t) = \prod_{i=1}^{k}(t - \lambda_i)^{n_i}$.

The Jordan form and its proof. For $k > 0$, the number of size k Jordan blocks for λ in the Jordan form for A is equal to

$$\dim\Big(\mathrm{Ker}(A-\lambda) \ \cap \ A^{k-1}\mathbb{K}^n\Big) \quad - \quad \dim\Big(\mathrm{Ker}(A-\lambda) \ \cap \ A^{k}\mathbb{K}^n\Big) \ .$$

Here are some specific things to expect on the exam.

- Several problems straight from or very close to the homework.
- At least one problem involving manipulations with Hermitian or scalar products, such as
  - If $< , >$ is a positive definite Hermitian matrix and $< Av, v >= 0$ for all v, then $A = 0$.
  - For a matrix A over $\mathbb{C}$, $< Av, v >$ is real for all v iff A is Hermitian.
  - If v is an eigenvalue of a symmetric matrix A over $\mathbb{R}$, then A maps the space of vectors orthogonal to v to itself.
  - If A is hermitian, then $(A - \lambda I)^2 v = 0$ implies $(A - \lambda I)v = 0$ (proof without appeal to spectra theorem – just do manipulations with $< , >$. (Note, given similarity to Jordan form, this result gives a fast proof that A is diagonalizable.)
- A problem using the Euclidean algorithm or a consequence, such as
  - Given two polynomials f, g with degree$(f) \geq$ degree(g), find polynomials q, r such that degree$(r) <$ degree(g) and $f = qg + r$.
  - Given two polynomials f, g, find a polynomial d which is a greatest common divisor of f and g
  - Suppose p is a nonzero irreducible polynomial in $\mathbb{K}[t]$, and p divides the product fg. Prove p divides f or p divides g.