MATH 405 – Scalar products on $\mathbb{R}^n$

A symmetric $n \times n$ real matrix defines a scalar product on $\mathbb{R}^n$ by the rule $(x, y) \mapsto x^{tr}Ay$. Every scalar product on $\mathbb{R}^n$ is defined in this way by a unique symmetric matrix.

The scalar product then defines a quadratic form $Q(x) = x^{tr}Ax$. For example,

$$
\begin{aligned}
Q(x) &= x^{tr}Ax \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
Q(x) &= (x_1, x_2) \begin{pmatrix} 2 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 2(x_1)^2 - 8x_1x_2 - 3(x_2)^2 \ .
\end{aligned}
$$

The quadratic form is a homogeneous degree two polynomial in the variables $x_1, \dots, x_n$. Here, "homogeneous" means that the polynomial is a sum of monomials of equal degree. Given such a polynomial Q, there is a unique symmetric matrix A such that $Q(x) = x^{tr}Ax$ for all x. When desired, we can write Q_A to indicate Q is defined by A in this way.

Suppose M is an invertible $n \times n$ real matrix. Then $Q_A(Mx) = Q_B(x)$, where $B = M^{tr}AM$. If $\mathcal{M}$ is the basis given by the columns of M, then B is the matrix which computes values for Q in $\mathcal{M}$-coordinates. To see this, let y denote $[x]_{\mathcal{M}}$, the coordinate vector for x with respect to the basis $\mathcal{M}$. The relation between y and x is simply that $My = x$. So,

$$
\begin{aligned}
y^{tr}By &= y^{tr}(M^{tr}AM)y \\
&= (My)^{tr}A(My) \\
&= x^{tr}Ax = Q(x) \ .
\end{aligned}
$$

Now suppose A is symmetric real, M is invertible and the matrix $M^{tr}AM$ is a diagonal matrix D. Sylvester's Theorem tells us that the number of positive/negative/zero entries on the diagonal of D only depends only on A (i.e., only on the scalar product, or equivalently only on the quadratic form), not on M. I will write η_-, η_0, η_+ below for the number of negative/zero/positive diagonal entries in D. The number of positive entries is called the index of positivity. The number of zero entries is called the index of nullity.

The proof of Sylvester's Theorem in Lang's book involved finding an "orthogonal" basis for $\mathbb{R}^n$ for the given scalar product (in coordinates of this basis, the scalar product is computed by a diagonal matrix), and then adding an argument that η_-, η_0, η_+ didn't depend on the choice of orthogonal basis.

It is convenient to be able to recognize/compute those indices easily from a given symmetric matrix A. Several facts can be used for this. Often it is easy to quickly see the indices for a 2×2 or 3×3 matrix.

- If A has a positive diagonal entry, then $\eta_+ \geq 1$.
  (If x is the ith canonical basis vector, then $x^{tr}Ax = A(i, i)$.)
- If A has a negative diagonal entry, then $\eta_- \geq 1$.
- $\eta_-/\eta_0/\eta_+$ is the number of negative/zero/positive roots of the characteristic polynomial of the symmetric matrix A, counted with multiplicity.
  - For example, if $\chi_A(t) = t^3(t-1)^7(t-5)^4$, then $\eta_- = 0, \eta_0 = 3, \eta_+ = 11$.
  - *In particular,* η_0 is the dimension of the null space (kernel) of A.

  This general fact holds because, given the symmetric matrix A, there is an orthogonal matrix U such that $U^tAU = D$, with D diagonal. The form "U^tAU" tells us we are just changing coordinates for the quadratic form. Because $U^{tr} = U^{-1}$, we have $U^tAU = U^{-1}AU$, and therefore A and D have the same characteristic polynomial.

- Suppose A is 2×2. Then the product of the two eigenvalues is the determinant of A, and the sum of the two eigenvalues is the trace of A (sum of the diagonal entries of A). So, $\det(A) < 0$ implies $\eta_- = \eta_+ = 1$. If $\det(A) > 0$, then the eigenvalues have the same sign, which is the sign of the trace.

Here is another trick (not necessary for our exams) you could use with larger matrices. Again, suppose A is symmetric real. Remember, for any invertible M, the matrix $B = M^{tr}AM$ (which must be symmetric) describes the same quadratic form in different coordinates. If $A(i,i)$ is nonzero, then we can choose M to add multiples of $A(i,i)$ to the other elements in row i to zero out those entries. The matrix M^{tr} acting on the other side of A will have the same effect on adding multiples of $A(i,i)$ to the other elements in column i. This lets us reduce to considering a smaller matrix.

For example, suppose $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 7 \\ 5 & 7 & 6 \end{pmatrix}$. Clearly $\eta_+ \geq 1$, but then what? Consider

$$\begin{aligned} M^{tr}AM &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 7 \\ 5 & 7 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & -3 \\ 3 & -3 & -19 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & -19 \end{pmatrix} . \end{aligned}$$

The symmetric matrix $\begin{pmatrix} 0 & -3 \\ -3 & -19 \end{pmatrix}$ has negative determinant. So for A we can conclude $\eta_- = 1, \eta_0 = 0, \eta_+ = 2$.