LIMINF and LIMSUP
for bounded sequences of real numbers

Definitions

Let $(c_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Define

$$\begin{aligned} a_n &= \inf\{c_k : k \geq n\} \ , \quad \text{and} \\ b_n &= \sup\{c_k : k \geq n\} \ . \end{aligned}$$

The sequence (a_n) is bounded and increasing, so it has a limit; call it a. This limit is by definition the liminf of the sequence (c_n),

$$\liminf_n c_n = \lim_{n\to\infty} \inf\{c_k : k \geq n\} \ .$$

Similarly, the sequence (b_n) is bounded and decreasing, so it has a limit; call it b. This limit is by definition the limsup of the sequence (c_n),

$$\limsup_n c_n = \lim_{n\to\infty} \sup\{c_k : k \geq n\} \ .$$

We have for all n that $[a_n, b_n]$ contains $[a_{n+1}, b_{n+1}]$. The intersection of these nested intervals is $[a, b]$.

The limit of a bounded sequence need not exist, but the liminf and limsup of a bounded sequence always exist as real numbers.

When there's no loss of clarity, we might omit the subscript variable (above, it is n). There are also shorter notations meaning the same thing: $\overline{\lim}\, a_n$ means $\limsup a_n$ and $\underline{\lim}\, a_n$ means $\liminf a$.

Associated facts

We continue with the above notations, and also let d_n denote a second bounded sequence of real numbers.

1. $[a_n, b_n]$ is the smallest closed interval which contains $\{c_k : k \geq n\}$.
2. $\lim c_n$ exists if and only if $\underline{\lim}\, c_n = \overline{\lim}\, c_n$. In this case $\lim c_n = \underline{\lim}\, c_n = \overline{\lim}\, c_n$.
3. There is a subsequence of (c_n) which converges to $\underline{\lim}\, c_n$.
4. There is a subsequence of (c_n) which converges to $\overline{\lim}\, c_n$.
5. $\underline{\lim}(c_n + d_n) \geq (\underline{\lim}\, c_n) + (\underline{\lim}\, d_n)$
(and strict inequality is possible).

6. $\overline{\lim}(c_n + d_n) \leq (\overline{\lim}\, c_n) + (\overline{\lim}\, d_n)$
   (and strict inequality is possible).

7. If α is a nonnegative real number,
   then $\overline{\lim}(\alpha c_n) = \alpha \overline{\lim}\, c_n$ and $\underline{\lim}(\alpha c_n) = \alpha \underline{\lim}\, c_n$

8. $\overline{\lim}(-c_n) = -\underline{\lim}\, c_n$ and $\underline{\lim}(-c_n) = -\overline{\lim}\, c_n$

9. (Putting the last two facts together:)
   If α is a negative real number,
   then $\overline{\lim}(\alpha c_n) = \alpha \underline{\lim}\, c_n$ and $\overline{\lim}(\alpha c_n) = \alpha \underline{\lim}\, c_n$

For an example of strict inequality in facts 5 and 6, consider $(c_n) = 0, 1, 0, 1, 0, 1, 0...$ (1 at even indices) and $(d_n) = 1, 0, 1, 0, 1, 0, 1, 0...$ (1 at odd indices).

Fact 8 is coming from the following fact for a bounded set S of real numbers, with $-S$ denoting $\{-s : s \in S\}$:

$$\inf(-S) = -\sup S \quad \text{and} \ \sup(-S) = -\inf S \ .$$

To get a feel for this, draw S and $-S$ for some example intervals: $S = [-3, -1]$, $S = [-3, 1]$, $S = [1, 3]$. Check the infs and sups of S and $-S$.

General sequences of real numbers.

Now drop the assumption that the sequence (c_n) must be bounded. Use the same definitions for liminf and limsup, but with $b_n = +\infty$ if $\{c_k : k \geq n\}$ is not bounded above, and with $a_n = -\infty$ if $\{c_k : k \geq n\}$ is not bounded below. (Note, $\underline{\lim}\, c_n = -\infty$ iff (c_n) is not bounded below, and $\overline{\lim}\, c_n = \infty$ iff (c_n) is not bounded above.) Now liminf and limsup are defined for all sequences of real numbers! Moreover, the "associated facts" still are true after just a little adjustment.

For fact 1, replace $[a_n, b_n]$ with $[a_n, b_n] \cap \mathbb{R}$ (in other words, if one of the endpoints is an infinity, remove it from $[a_n, b_n]$).

Generalize the interpretation of "limit exists" (converges) to include the cases where the limit is ∞ or $-\infty$ (where we have already a definition of what e.g. $\lim c_n = \infty$ means).

For facts 5 and 6, generalize "+" to include, for every real number α: $\infty + \alpha = \infty$; $-\infty + \alpha = -\infty$. Also define $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$. With these definitions, the facts 5 and 6 are still true. The one thing we don't do is define $-\infty + \infty$. Knowing only $\underline{\lim}\, c_n = -\infty$ and $\overline{\lim}\, d_n = \infty$ gives us zero information about $\underline{\lim}(c_n + d_n)$ or $\overline{\lim}(c_n + d_n)$.