

## Open sets, closed sets and sequences of real numbers

**Definition.** The distance between real numbers  $x$  and  $y$  is  $|x - y|$ .

For example,  $\text{dist}(-4, 3) = |(-4) - (3)| = 7$ .

**Definition.** For a real number  $x$  and  $\epsilon > 0$ ,

$$B_\epsilon(x) = \{y \in \mathbb{R} : \text{dist}(x, y) < \epsilon\}.$$

Of course,  $B_\epsilon(x)$  is another way of describing the open interval  $(x - \epsilon, x + \epsilon)$ . We also call this an epsilon neighborhood of  $x$ .

**Definition.** An *open* subset of  $\mathbb{R}$  is a subset  $E$  of  $\mathbb{R}$  such that for every  $x$  in  $E$  there exists  $\epsilon > 0$  such that  $B_\epsilon(x)$  is contained in  $E$ .

For example, the open interval  $(2, 5)$  is an open set. Any open interval is an open set. Both  $\mathbb{R}$  and the empty set are open. The union of open sets is an open set.

The complement of a subset  $E$  of  $\mathbb{R}$  is the set of all points in  $\mathbb{R}$  which are not in  $E$ . It is denoted  $\mathbb{R} \setminus E$  or  $E^c$ . (Note, the second notation requires you to know that the complement is defined relative to  $\mathbb{R}$ .)

$[2, 5]$  is not an open set, but its complement  $(-\infty, 2) \cup (5, \infty)$  is open. This is an example of the next which provides a useful perspective on convergence.

**Proposition C.** Suppose  $E$  is a subset of  $\mathbb{R}$ . The following are equivalent.

1. If a sequence of points from  $E$  converges, then the limit of the sequence is in  $E$ .
2. The complement of  $E$  is an open set.

**Names.** There are names for the two properties listed above:

1.  $E$  is sequentially closed.
2.  $E$  is closed.

In your future, you may see the more general situations in which the two notions need not be the same. In  $\mathbb{R}$ , you can use either to mean “closed”.

For the next summary, here is some notation. A sequence *converges in  $E$*  if it converges and its limit is in  $E$ . A *cover* of a set  $E$  is a collection  $\mathcal{C}$  of sets whose union contains  $E$ . A subcover is a collection of some of the sets in  $\mathcal{C}$  whose union still contains  $E$ . A finite subcover is a subcover which uses only finitely many of the sets in  $\mathcal{C}$ . An *open cover* is a cover by a collection of sets all of which are open.

**Proposition K.** Suppose  $E$  is a subset of  $\mathbb{R}$ . The following are equivalent.

1.  $E$  is closed and bounded.
2. Every sequence from  $E$  has a subsequence which converges in  $E$ .
3. Every open cover of  $E$  has a finite subcover.

**Names.** The last two properties have names:

- (2)  $E$  is sequentially compact.
- (3)  $E$  is compact.

In  $\mathbb{R}$  the conditions are equivalent, and you can use “compact” and “sequentially compact” interchangeably.

The condition (3) is more abstract. You’re not required to know about it now. But you should be able to prove the other two conditions are equivalent. As follows, for example.

## Proof

(1)  $\implies$  (2)

Suppose  $E$  is closed and bounded. Suppose  $(a_n)$  is a sequence of points from  $E$ . Since it is bounded, it has a convergence subsequence. Since  $E$  is closed, the limit is in  $E$ . Therefore  $E$  is sequentially compact.

(2)  $\implies$  (1)

It suffices to show that if  $E$  is not closed and bounded, then  $E$  is not sequentially compact.

If  $E$  is not closed, then there is a sequence  $(a_n)$  of points from  $E$  which converges to a point outside  $E$ . Since every subsequence of  $(a_n)$  has the same limit, no subsequence of  $(a_n)$  can converge in  $E$ . Therefore  $E$  is not sequentially compact.

If  $E$  is not bounded, then there for each  $n \in \mathbb{N}$  we may pick a point  $a_n$  from  $E$  such that  $|a_n| > n$ . Then no subsequence of  $(a_n)$  is bounded, so no subsequence of  $(a_n)$  can converge, and therefore  $E$  is not sequentially compact.

This finishes the proof.