MATH 475 – Spring 2005 – FINAL EXAM (take-home part)

INSTRUCTIONS. You may use books (although they're not necessary), but you may not consult with other people.
When the answer is a number, compute the number, not only a formula in numbers – e.g., from 5! – 3!, compute 114.
Each problem is worth 15 points.

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1. Let π be a random permutation π of the set $\{1, 2, \ldots, n\}$. Let p_n denote the probability that π has a cycle of length at least $n/2$.
   (a) Compute p_n.
   (b) Compute $\lim_{n\to\infty} p_n$.
   (c) Does your formula in part (a) work if $n/2$ is replaced by $n/3$? Explain.

2. Let G be the directed graph with adjacency matrix $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$. (Here $A(i, j)$ is the number of edges in G from vertex i to vertex j.) Let p_n denote the number of paths of length n from vertex 1 to vertex 2 (these are the paths of n edges beginning at vertex 1 and ending at vertex 2).
   Compute $\lim_{n\to\infty} 7^{-n} p_n$.

3. A permutation is an *involution* if every cycle has length 1 or length 2. (In other words, composing the permutation with itself gives the identity permutation.) Let a_n denote the number of permutations of $\{1, 2, \ldots, n\}$ which are involutions.
   (a) Find a recurrence relation for the sequence (a_n).
   (b) What is the probability that an involution of $\{1, 2, 3, 4, 5, 6\}$ fixes 1?
   (By definition, this is b_6/a_6, where b_6 is the number of involutions π of $\{1, 2, 3, 4, 5, 6\}$ such that $\pi(1) = 1$.)

4. Four married couples have dinner around a circular table. How many ways can they be seated such that spouses are never adjacent?

5. How many ways are there to divide 6 baseballs, 6 footballs, 6 volleyballs, 6 tennis balls and 6 beach balls between two brothers, so that each gets 15 balls?

6. Let G be a graph with n vertices and more than $(n - 1)(n - 2)/2$ edges. Prove that G is connected. (Note, as G is a graph, for all vertices i and j, there is no edge from i to i, and there is at most one edge from i to j.)

7. Suppose a sphere is tiled by pentagons and hexagons subject to the following conditions: where two of the shapes meet, their intersection is a common vertex or a common edge; and each vertex lies on exactly three edges.
   Prove that there is only one possibility for the number of pentagons used in such a tiling, and compute that number. (Hint: to check, look at a soccer ball.)