Struggles of the Past: How New Types of Numbers Emerged in the History of Mathematics

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I. NUMBERS

"What is a number?"

We take a lot of our knowledge of numbers for granted. So much seems easy – counting, zero, negative numbers, fractions, real numbers, etc. What's the problem!

But, these were hard-won advances over millenia. Even brilliant mathematicians (e.g. Brahmagupta, Leibniz, Euler) made mistakes in struggles with new numbers. [Perhaps an example could comfort a student?] Even after successful introduction, there were often long delays before the advances were fully accepted and used.

Also, in the history of numbers we can see some pro's and con's of the stereotypical "physicist's approach" and "mathematician's approach".

"The physicist" grabs a mathematical idea which seems to help and goes with it. Those bean counter mathematicians will get around to crossing the t's and dotting the i's.

"The mathematician" cares for Truth and struggles to produce definitions and arguments which are clear, sensible and reliable. So what if nonsense works – it is still nonsense!

In the end we need both approaches. Mathematics is like a tree, with roots and branches, and both keep growing.

1. COUNTING

There is a famous quote of Kronecker (1823-1891): "God created the integers, all else is the work of man."

I disagree!

Of all the great steps forward with numbers, I'm most impressed by the conception of the abstract counting numbers 1,2,3,4,... .

It was not automatic to think of two doves and two wolves as having something in common. A counting number makes a conceptual Thing out of the process of being able to put two sets into one to one correspondence. Not all cultures have had a systematic way of naming arbitrarily large numbers. We have not observed this capability in a nonhuman species.

There is some history of the evolution of the idea of counting numbers, with reasoned speculation on how this capability emerged in human evolution, e.g.

- "Number, the language of science", Tobias Dantzig
- "Number words and Number Symbols", Karl Menninger
- "The Math Gene", Keith Devlin .

2. NEGATIVE NUMBERS

The Babylonians around 2000 BC were solving many problems which boiled down to solving a quadratic equation. But their problems were rooted in geometry and the material world, and they ignored negative number solutions.

The ancient Greeks likewise considered only positive number solutions. (The great Greek mathematician Diophantus, 3rd century AD, in "Arithmetica" referred to a negative solution of a problem as being "absurd".) For those Greeks, numbers were attached to explicit counting and ratios.

By 200 BC, in China, red and black counting rods were used for money held and owed.

By the Han dynasty (circa 200BC-200AD) there were written Chinese works with rules for manipulating negative quantities appearing in calculations (for problems which had positive solutions).

Circa 620 A.D. Brahmagupta in India essentially wrote down rules of arithmetic for negative and positive numbers (e.g. "the product or quotient of a fortune and a debt is a debt").

By the 1300's such rules had appeared in writing in Europe.

Cardano's Ars Magnus (1545) showed a clear acceptance of negative numbers as possible solutions to equations.

But even in the 18th century (!), there was still some resistance to negative numbers. The British mathematician Francis Maseres wrote (1759) that negative numbers
"*darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple*".

Maseres was probably heretical among mathematicians for this view – but still, he was made a Fellow of the Royal Society in 1771, over 100 years after Newton invented calculus.

Much ado about nothing

3. ZERO

It was hard to invent counting numbers. Much harder still to make a Thing which is nothing – i.e. to invent zero.

Zero appeared in numbers as a placeholder in place value systems. E.g. in our base 10 system, in 7105, the "0" means we add no multiple of 10: $7105 = 7(1000) + 1(100) + 5(1)$. But it is a big extra step to make zero a Thing and to integrate zero into arithmetic.

By the 7th century AD, the Hindus had begun to recognize "sunya", the absence of quantity, as a quantity in its own.right. I.e., zero was treated as a number.

Brahmagupta (7th century A.D.), was the first known mathematician to try to extend the rules of arithmetic to include zero. His rules for arithmetic involving zero were fine, apart from the problem of defining division by zero. There he and his Hindu successors had a heck of a time.

4. TRYING TO DEFINE DIVISION BY ZERO

• Brahmagupta (7th century A.D.) ” *A zero divided by a zero is zero*” and ”*a negative or a positive divided by zero has that zero as its divisor*”.

• Mahavira (9th century A.D.)

“A number remains unchanged when divided by zero.”

• Bhaskara (II) (12th century A.D.)

"A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity. In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth."

(He seems to be saying that n/0 = infinity. But that would mean that n = 0 x infinity , for every number n ...)

In his text, Bhakskara (II) posed problem of finding an unknown number "... whose multiplier is 0. Its own half is added. Its multiplier is 3; its divisor 0. The given number is 63."

We can translate this problem as: Find the number x such that

$$3\left(\frac{0x + (1/2)0x}{0}\right) = 63 \ .$$

It seems we are supposed to cancel 0's (!) to get $3(x + (1/2)x) = 63$ and then solve to get $x = 14$.

The correct resolution of these miseries is to give up. It's impossible to give a satisfactory definition of division by zero – so don't try.

5. WHY CAN'T WE DIVIDE BY ZERO?

This is a case where an axiomatic approach is crucial.

We write down some properties of arithmetic we consider essential. These are assumptions. Then we show it's impossible to define division by zero and keep these properties. I'll write $\times$ for multiplication.

One consequence of the properties is that zero times any number is zero, because

$$a \times 0 = a \times (0 + 0) = (a \times 0) + (a \times 0)$$

and then after subtracting $a \times 0$ from both sides we conclude $0 = a \times 0$.

Next, however we consider division by a number b, we insist that it reverses multiplication by b:

$$(a \times b) \div b \;=\; a \quad .$$

We choose to regard this as an essential property of division. Multiplying by b and then dividing by b doesn't change a number. For example,

$$\begin{aligned} (3 \times 2) \div 2 \;&=\; 3 \\ (-5 \times 2) \div 2 \;&=\; -5 \\ (0 \times 2) \div 2 \;&=\; 0 \end{aligned}$$

$$(a \times 2) \div 2 \;=\; a \quad \text{for any number } a \; .$$

So if we can define division by zero, then for any number a,

$$\begin{aligned}(a \times 0) \div 0 &= a \\ (0) \div 0 &= a\ .\end{aligned}$$

Therefore every number equals $(0) \div 0$ (whatever that is). Therefore all numbers are equal. That is a contradiction.

So we can't define division by zero without losing the property $(a \times b) \div b = a$.

Score one for "the mathematicians". It is useful to know what is impossible (rather than continuing to try ...).

By the way: in a careful modern construction of numbers, there are just TWO fundamental operations, addition and multiplication. Subtraction and division are not fundamental.

In the end, the ONLY property connecting addition and multiplication is the distributive law:

$$\begin{aligned} a \times (b + c) &= (a \times b) + (a \times c) \\ (a + b) \times c &= (a \times c) + (b \times c) \ . \end{aligned}$$

5. IS ZERO AN EVEN NUMBER?

Ask, “What is the key to love?”

A philosopher or poet replies with many words.

A mathematician replies,
“what is your definition of love?”

After a while, you realize you can’t answer some questions without definitions.

To make a claim about something, you want to know what it is you are talking about.

Definition. An even number is an integer n such that there is an integer k such that $n = 2 \times k$.

Examples:

$$
\begin{aligned}
n &= 2 \times k \\
\\
-4 &= 2 \times (-2) \\
-2 &= 2 \times (-1) \\
0 &= 2 \times 0 \\
2 &= 2 \times 1 \\
4 &= 2 \times 2 \quad \ldots
\end{aligned}
$$

Zero is an even number.

6. "HARRIOT'S PRINCIPLE"

Before moving on let's appreciate that "zero" allows a useful technique for solving polynomial equations. It was proposed by Thomas Harriot in the early 1600s and popularized later by Descartes. (The author Tobias Dantzig referred to this technique as "Harriot's Principle".)

The technique: subtract a polynomial from both sides to make one side zero. (This does not change the solution set.) Then find a root of the resulting polynomial.

Example. Solve

$$9x^5 + 3x^2 + 2x - 5 \;=\; 9x^5 + 2x^2 - 2$$

Subtract the right side from both sides:

$$\begin{aligned}(9x^5 + 3x^2 + 2x - 5) - (9x^5 + 2x^2 - 2) \;&=\; 0\\ x^2 + 2x - 3 \;&=\; 0\end{aligned}$$

Much easier ...

This method is a triviality today, but it was a great leap forward back in the day.

It is another example of how we can forget the difficulty of progress (until we see our students struggle?).

7. THOSE RATIONAL ANCIENT GREEKS

The remarkable ancient Greeks thought very carefully about mathematics. They introduced formal axioms and theorems.

A Greek would think of a magnitude as perhaps the length of a stick. Two sticks would have commensurable lengths if, for example, 3 copies of one stick would give the same total length as 5 copies of the other.

More formally, suppose x and y are the magnitudes (lengths of the sticks). They would be commensurable if $3x = 5y$. Or if $8x = 7y$. Or if there are any positive integers such that $px = qy$.

Another way to say $px = qy$ is to say $x = (p/q)y$. That is, y is a rational number multiple of x. In particular: x and 1 are commensurable if and only if x is a rational number.

These Greeks thought of numbers as commensurable magnitudes. The Pythagoreans circa 550 BC thought number was the basis for everything: “all is number”. So they naturally felt all magnitudes should be commensurable.

But by 400 BC, they had learned this is false. (For example, in a square, the lengths of a side and the diagonal are not commensurable. Their ratio is $\sqrt{2}$.)

For those Greeks, this was a dramatic fact of great philosophical import.

8. A PROOF THAT THE SQUARE ROOT OF 2 IS NOT A RATIONAL NUMBER.

Suppose $\sqrt{2} = p/q$, with p and q positive integers. After cancelling 2's as much as possible, we can assume also that at least one of the numbers p and q is an odd integer.

Since $\sqrt{2} = p/q$, after squaring we get $2 = p^2/q^2$, and therefore $2q^2 = p^2$.

This tells us p^2 is divisible by 2. If p were odd, then p^2 would be odd; so, p is even.

Now, because p is even, we can divide the equation $2q^2 = p^2$ by 2 and get that $q^2 = p^2/2 = (p/2)p$. Since p is even, this tells us that q^2 is even, and again that implies q is even.

Therefore both p and q are even. That is a contradiction. So our original assumption that we could write $\sqrt{2} = p/q$, with p and q positive integers, must be wrong.

Therefore $\sqrt{2}$ is irrational.

9. THE REAL NUMBERS

Today, we have an intuitive idea of a real number as indicating a point on a number line. Intuitively, a positive real number x indicates a point at a certain distance (or magnitude) to the right of zero, and a negative number indicates a point at a certain distance to the left of zero. We multiply, add, etc. these magnitudes with impunity. The existence of irrational numbers just means that some of these magnitudes aren't commensurable. No problema.

Euclid thought of numbers as magnitudes commensurable with some given unit. If we take that unit to be the number 1, then those magnitudes correspond to the postive rational numbers. Euclid did not define an arbitrary magnitude as a number and he didn't multiply incommensurable magnitudes. (What would it mean?)

This careful distinction between magnitude and number turned out to be unnecessary. Eventually Islamic mathematicians (specifically, abu Kamil ibn Aslam, circa 900 AD) did useful arithmetic with irrational numbers given by expressions like $\sqrt{1/2+\sqrt{5/4}}$ and simply disregarded the cautions of Euclid. No problema. (Score one for the physicists.)

Only in the 1800s–long after calculus was invented–was there at last a satisfactory rigorous development of the real numbers and their arithmetic.

On to the imaginary

10. THE COMPLEX NUMBERS

Square roots of negative numbers ("imaginary numbers") show up when one tries to solve polynomial equations. For example $x^2 = -1$, or equivalently $x^2 + 1 = 0$.

Already the Babylonians by 2000 BC were essentially solving quadratic equations. You might think that they would have discovered complex numbers. Not so. For them, solutions were positive numbers.

A complex number is (informally) a number such as $3 + 2\sqrt{-5}$, a real number plus a real multiple of an imaginary number. Complex numbers appeared in print in Cardano's "Ars Magnus" (The Great Work) (1545 AD). (Cardano's summation in the book: *"Written in 5 years, may it last as many thousand".*)

Cardano's book contained a problem: divide 10 into parts such that the product is 40. If one of those parts is named x, then the other part is $10 - x$, and that number x must satisfy the equation

$$x(10 - x) = 40 \ .$$

Cardano solved this equation. The solutions are complex numbers; they are not real numbers. He wrote

"So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless."

11. COMPLEX NUMBERS AND THE CUBIC

Cardano considered these numbers because his book gave a solution to the ancient problem of finding a root of a cubic polynomial with real coefficients. In Cardano's formula, square roots of possibly negative numbers appeared, but nevertheless one could produce from the formula roots which were real numbers.

For the equation $x^3 = 15x + 4$, Cardano's formula for a solution gave

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}\ .$$

But how does that lead to the solution $x = 4$?

Let us simplify notation a bit and use the symbol i to denote a new number (i.e., a number which is not a real number) whose square is -1.

Then in Cardano's formula

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

we could replace $\sqrt{-121}$ with $(\sqrt{121})i = 11i$ and get

$$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}\ .$$

You can compute $(2+i)^3 = 2+11i$ (just do it!) and likewise $(2-i)^3 = 2-11i$. Then we can think of the formula as giving $x = (2+i)+(2-i) = 4$. We can also get the solutions $-2+\sqrt{3}$ and $-2-\sqrt{3}$ by finding different cube roots of $2-11i$.

Cardano didn't understand these "fictitious" numbers very well; but he used them to good effect. Score another for "the physicists" ...

12. TROUBLE WITH COMPLEX ROOTS

Before the 1800s (with Gauss, Cauchy, Riemann ...) understanding of complex numbers developed slowly. Euler (1707-1783) introduced the symbol i for $\sqrt{-1}$ and found the magic formula $e^{i\pi} + 1 = 0$.

Even Euler had some trouble. At one point Euler wrote that the general rule for square roots (he meant the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$) shows

$$\sqrt{-1}\sqrt{-4} = \sqrt{(-1)(-4)} = \sqrt{4} = 2 \ .$$

But this reasoning leads to contradictions such as

$$-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \ .$$

Evidently that "general rule" cannot be extended from the positive real numbers to the complex numbers.

The problem has to do with thinking clearly about roots. The words "i is the square root of -1" contain the trap. If $i^2 = -1$, then the square of $-i$ is also -1. But "the" square root of -1 suggests there is only number whose square is -1.

We get the complex numbers right by introducing just one special number i (which satisfies $i^2 = 1$) and then letting the complex numbers be combinations $a + ib$, where a and b are real numbers.

Now every nonzero complex number will have exactly 2 square roots and exactly 3 cube roots. For example, to find the three cube roots of 1, you can solve $x^3 - 1 = 0$ by factoring $x^3 = 1 = (x-1)(x^2+x+1)$ and then using the quadratic formula on the second factor.

So the cube root in Cardano's formula is not well defined! No wonder his formula was confusing.

Even “a physicist” benefits from the clarity achieved by the precise formal definition of the complex numbers. Sometimes it does help to make sense.

13. THE GEOMETRIC INTERPRETATION OF COMPLEX NUMBERS

A big step toward selling and understanding the complex numbers was their geometric interpretation as elements in the plane, and the geometric interpretation of complex addition and multiplication. This only happened around 1800 (!), in publications of Wessel (1797) and Argand (1806). We consider the complex numbers as numbers of the form $a+ib$, where a and b are real numbers. Then we picture the numbers $a+ib$ as the points (a,b) in the plane.

For an exposition of the basic properties of complex numbers, with connection to calculus, and exercises for students, see the notes of myself or Professor Hamilton on our departmental course page for MATH 141 at
www-math.umd.edu/undergraduate/courses/

14. THE FUNDAMENTAL THEOREM OF ALGEBRA

Today the complex numbers are indispensable to mathematics. One reason is the Fundamental Theorem of Algebra (FTA): every nonconstant polynomial has a root which is a complex (possibly real) number. (A fully correct proof was not given until the 1800's). An equivalent statement is that a nonconstant polynomial can be written as a product of factors of the form (x-a), where a is a complex number (possibly real). (Equivalent because $(x-a)$ is a factor if and only if a is a root.) For example,

$$\begin{aligned} & x^5+x^4+x^3+x^2 \\ =\ & x^2(x+1)(x^2+1) \\ =\ & (x-0)(x-0)(x-[-1])(x-i)(x+i)\ . \end{aligned}$$

Leibniz was smart enough to invent the calculus (independently of Newton). Even Leibniz had troubles here.

Leibniz claimed (for a little while) that for a positive real number a, the polynomial $x^4 + a^4$ has no complex number root.

Let us use $a = 1$ as an example here, that is we consider the polynomial $x^4 + 1$. Leibniz noted that a solution would be $x = \sqrt{\sqrt{-1}}$. He thought this could not be a number of the form $a + b\sqrt{-1}$ (with a and b real numbers).

Leibniz was wrong. Problem: find a complex number x such that $x^4 + 1 = 0$.

This again shows "$\sqrt{-1}$" is not very good for a definition.

15. ADDING UP INFINITELY MANY NUMBERS

Suppose $a_1, a_2, \ldots$ are nonnegative numbers. Then $a_1 + a_2 + \ldots$ will either be infinity or a nonnegative real number (called the "sum" of the the infinite series $a_1 + a_2 + \ldots$).

Many infinite series were "summed" long ago, for example

- ("Zeno")
  $1/2 + (1/2)^2 + (1/2)^3 + (1/2)^4 + \cdots = 1.$

- (Oresme, ca. 1350)
  $(\frac{1}{2})1 + (\frac{1}{4})2 + (\frac{1}{8})3 + (\frac{1}{16})4 + \cdots = 2$.

- $(1/4) + (1/4)^2 + (1/4)^3 + \cdots = 1/3$.

The last three sums can be explained with simple pictures.

The next one is harder.

- (Nilikantha $\sim$ 1500; Madhava 1300s?; Leibniz, Gregory 1670s)
  $1 - (1/3) + (1/5) - (1/7) + \cdots = \pi/4$

After a while summing series is as natural as adding up finitely many numbers.

16. NUMBER AND THE FOUNDATIONS OF PROBABILITY

In probability, if you have a (possibly infinite) list of disjoint (nonoverlapping) events

$$A_1, A_2, A_3, \ldots$$

and if A is the union of all these events,

then it is a fundamental axiom that the probability of the union is the sum of the probabilities:

Prob(A) = Prob(A_1) + Prob (A_2) + ...

Example. An immortal works on a problem. Let A_n be the event he succeeds on day n. Suppose Prob(A_n) = $(1/4)^n$.

Now let A be the event that the immortal succeeds at all, i.e. on some day. Then A is the union of the nonoverlapping events A_n, and

$$\text{Prob}(A) = 1/4 + (1/4)^2 + (1/4)^3 + ... = 1/3 \ .$$

On the other hand, if in the example Prob(A_n) = 0 for every n, then Prob(A) $= 0 + 0 + \cdots = 0$.

If the immortal has probability zero of solving the problem on every nth day, well, he has probability zero of solving the problem at all.

Now suppose you are explaining probability in your stat class.

You tell a student that if you pick a number randomly from the unit interval [0,1], and if $0 \leq a < b \leq 1$, then the probability of the number being in $[a, b]$ equals $b - a$. (For example, the probability of the number being between 1/3 and 1/2 is 1/6.)

In particular, the probability of picking some number in $[0, 1]$ is 1 (of course!) and the probability of picking any particular number is zero.

But suppose your student is smart, and objects:

This is ridiculous! Consider an infinite list of all the numbers in $[0,1]$. Say A_n is the event that the nth number on the list is picked.

Let A be the event that some number from $[0,1]$ is picked. You said Prob$(A) = 1$.

Since we have listed all the numbers, the event A that a number gets picked is the union of the events A_n. And since the probability of picking any particular number is zero, that fundamental axiom would say

$$
\begin{aligned}
1 &= \text{Prob}(A) \\
&= \text{Prob}(A_1) + \text{Prob}(A_2) + \ldots \\
&= 0 + 0 + \ldots \\
&= 0\ .
\end{aligned}
$$

Contradiction ...

What's wrong with your student's argument?

To answer that we need cardinal numbers.

17. CARDINAL NUMBERS

The counting numbers describe the sizes of finite sets. They are a special case of "cardinal numbers", which describe the sizes of sets, which might be finite or infinite.

Two sets are defined to have the same cardinality (size) if the elements inside them can be put into one to one correspondence.

Example: the set of fingers on my left hand and the set of fingers on my right hand have the same cardinality (5).

Example of Galileo: the set of positive integers and the set of even positive integers have the same cardinality ! The rule $n \mapsto 2n$ gives the one-to-one correspondence:

1, 2, 3, 4, 5, 6, ...
2, 4, 6, 8, 10, 12 ... .

A set is called countable if it is finite or has the same cardinality as the set of positive integers 1,2,3,... .

A set of numbers is countable if and only if it is possible to make an infinite list of the numbers from the set such that every number is on the list.

The set of all integers is countable.
Here's a list: 0,1,-1,2,-2,3,-3, . . .
Clearly every integer appears on this list, exactly once.

The set of rational numbers is also countable. This is a little harder to show (next page)

Here's one way to make a list of all the rational numbers:

-1, 0, 1,

-2, 2,
-3/2, -1/2, 1/2, 3/2,

-3, 3,
-5/2, 5/2,
-8/3, -7/3, -5/3, -4/3, -2/3, -1/3, 1/3, 2/3, 4/3, 5/3, 7/3, 8/3,

and so on. At the nth stage, we are adding to the growing list all the rational numbers in $[-n, n]$ which can be written with denominators $1, 2, \ldots, n$, and which are not yet on the list. Eventually every rational number gets on the list.

BUT! Cantor's genius was to conceive of and prove the following (late 1800s):
Not all infinite sets are countable. In particular:

The interval [0,1] is not countable.

and this is the problem with your good student's argument. That fundamental axiom of probability tells you that the probababability of the sum is the sum of the probabilities for COUNTABLY MANY disjoint events. There is no axiom for adding up probabilities of uncountably many events. We just have to stay away from that, just as we have to stay away from division by zero.

18. Proof that [0,1] is uncountable.

We prove the unit interval [0,1] is not countable by the method of contradiction, using "Cantor's Diagonal Argument".

For [0,1] to be countable would mean that there is some (infinite) listing of numbers from [0,1] such that every number in [0,1] appears on the list.

So suppose we have such a listing of all the numbers in [0,1]. All we have to do is exhibit a number from [0,1] which is not on the list. That contradiction proves no such list exists.

For example suppose we have (in decimal notation) an infinite list of numbers $x_1, x_2, \ldots$ from $[0, 1]$:

$$
\begin{aligned}
x_1 &= .\boxed{6}24453\ldots \\
x_2 &= .3\boxed{5}8711\ldots \\
x_3 &= .33\boxed{4}229\ldots \\
x_4 &= .458\boxed{7}28\ldots \\
x_5 &= .0015\boxed{2}2\ldots \\
x_6 &= .22756\boxed{6}\ldots
\end{aligned}
$$

and so on. We just pick a number y such that the nth digit in its decimal represenation is not the nth digit of x_n. (I put boxes around the numbers x_n for easier viewing). For example, take $y = .448488\ldots$. Now y can't be on the list, because it can't be any of the x_n. QED.

(This is called the "diagonal argument" because those numbers x_n lie on some diagonal line.)

19. CONCLUSION.

So what is a number? There are lots of numbers, including more we haven't considered. Quaternions, ordinal numbers, surreal numbers, algebraic numbers ... They have various different meanings.

In all cases we have two ingredients.

- **Rigor.** We have symbols and rules for manipulating them which are allowed to produce proofs. This is rigor. In principle a mindless machine can check the proof by checking if each manipulation is legal.

- **Meaning.** We have – in some "Platonic reality" – in some shared dream world of human experience – an idea of what those symbols and rules mean.

All the kinds of numbers have the same basic status for rigor (scribblings a machine could check), and for meaning, as fragments in the evolving human imagination.

Some kinds of numbers are easier for us than others. But just as you don't see $\sqrt{-1}$ out your window, so also you don't see an infinite set of counting numbers. Let alone uncountably many real numbers. You don't even see the number 5: this is an abstraction about two sets being in one to one correspondence.

The "unreasonable effectiveness of mathematics" is the amazing fact that our scribblings and dream world have such powerful application in the real world.

One more number ...

The number on the clock means my time is up.