PERIODIC POINTS FOR ONTO CELLULAR AUTOMATA

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Summary. Let φ be a one-dimensional surjective cellular automaton map. We prove that if φ is a closing map, then the configurations which are both spatially and temporally periodic are dense. (If φ is not a closing map, then we do not know whether the temporally periodic configurations must be dense.) The results are special cases of results for shifts of finite type, and the proofs use symbolic dynamical techniques.

1. Introduction and sermon

Let φ be a surjective one-dimensional cellular automaton map (in the language of symbolic dynamics, φ is a surjective endomorphism of a full shift). Must the set of φ-periodic points be dense? This is a basic question for understanding the topological dynamics of φ, and we are unable to resolve it.

However, if φ is a closing endomorphism of a mixing subshift of finite type σ_A, then we can show the points which are periodic for both φ and σ_A are dense (Theorem 4.4). This is our main result, and of course it answers our question in the case that the c.a. map is closing. We give a separate proof for the special case that φ is an algebraic map (Proposition 3.2). The proofs are completely different and both have ingredients which might be useful in more general settings.

The paper is organized so that a reader with a little background can go directly to Sections 3 and 4 and quickly understand our results.

We work in the setting of subshifts of finite type, and to explain this to some c.a. workers we offer a few words from the pulpit. Dynamically, one-dimensional cellular automata maps are best understood as particular examples of endomorphisms of mixing subshifts of finite type. The resources of this setting are needed to address some c.a. questions, even if one cares not at all about the larger setting. But, one should. Even apart from other motivations, the setting of subshifts of finite type (rather than just the full shifts of the c.a. case) is philosophically the right setting for c.a. A c.a. map is a locally determined rule of temporal evolution; allowing shifts of finite type as domains simply allows local conditions on spatial structure as well. This is natural "physically" and unavoidable dynamically: a cellular automaton is usually not surjective, and usually the possible spatial configurations after an iterate are no longer those of a full shift.

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2. Definitions

Let S be a finite set of n elements, with the discrete topology. Let Σ_n be the product space $S^{\mathbb{Z}}$, with the product topology. We view a point x in Σ_n as a doubly infinite sequence of symbols from S, so $x = \dots x_{-1}x_0x_1\dots$. The space Σ_n is compact, metrizable and one metric compatible with the topology is $\text{dist}(x, y) = 1/(|n|+1)$ where $|n|$ is the minimum nonnegative integer such that $x_n \neq y_n$. The shift map $\sigma : \Sigma_n \to \Sigma_n$ is the homeomorphism defined by the rule $\sigma(x)_i = x_{i+1}$. The topological dynamical system (Σ_n, σ) is called the *full shift on n symbols* (S is the symbol set). If X is a nonempty compact subset of Σ_n and the restriction of σ to X is a homeomorphism, then $(X, \sigma|_X)$ is a *subshift*. (We may also refer to either X or $\sigma|_X$ as a subshift, also we may suppress restrictions from the notation.) Equivalently, there is some countable set $\mathcal{W}$ of finite words such that X equals the subset of Σ_n in which no element of $\mathcal{W}$ occurs. A subshift (X, σ) is a *subshift of finite type* (SFT) if it is possible to choose a finite set to be a defining set $\mathcal{W}$ of excluded words. The SFT is k-step if there is a defining set W with words of length at most $k+1$.

If A is an $m \times m$ matrix with nonnegative integral entries, let G_A be a directed graph with vertex set $\{1, \dots, m\}$ and with $A(i, j)$ edges from i to j. Let E_A be the edge set of G_A. Let Σ_A be the subset of $(E_A)^{\mathbb{Z}}$ obtained from doubly infinite walks through G_A; that is, a bisequence x on symbol set E_A is in Σ_A if and only if for every i in $\mathbb{Z}$, the terminal vertex of the edge x_i equals the initial vertex of the edge x_{i+1}. Let $\sigma_A = \sigma|_{\Sigma_A}$. The system (Σ_A, σ_A) (or Σ_A or σ_A) is called an *edge shift*, and it is a one-step SFT.

Let X_A be the space of one-sided sequences obtained by erasing negative coordinates in Σ_A: that is, if a point x is in Σ_A, then the one-sided sequence $x_0x_1x_2\dots$ is in X_A, and X_A contains only such points. The shift map rule $\sigma(x)_i = x_{i+1}$ defines a continuous surjection $X_A \to X_A$, also called σ_A (by abuse of notation). Except in the trivial case that X_A is finite, this map σ_A is only a local homeomorphism. The system (X_A, σ_A) is a *one-sided subshift of finite type*. The proof of our main result argues by way of the one-sided SFTs.

An SFT is called irreducible if it has a dense forward orbit. A nonnegative matrix A is irreducible if for every i, j there exists $n > 0$ such that $A^n(i, j) > 0$, and it is primitive if n can be chosen independent of (i, j). An irreducible matrix A defines an edge shift which is an irreducible SFT, and a primitive matrix A defines an edge shift which is a mixing SFT. For any A, if B is a maximal irreducible principal submatrix of A, then we can view the edge set E_B as a subset of E_A, and the edge shift X_B is an *irreducible component* of X_A. X_B is a *terminal* irreducible component if there is no path in G_A from E_B to a point in another irreducible component.

A homomorphism φ of subshifts is a continuous map between their domains which commutes with the shifts. A factor map is a surjective homomorphism of subshifts. There are two distinct types of factor maps between irreducible SFT's. If there is a uniform bound to the number of preimages of each point the factor map is called *finite-to-one*. If there is no uniform bound the map is called *infinite-to-one*. Under an infinite-to-one factor map "most" points will have uncountably many preimages. A topological conjugacy or isomorphism of subshifts is a bijective factor map between them. If there is an isomorphism between two subshifts, then they are topologically conjugate, or isomorphic. Any SFT is topologically conjugate to some edge SFT.

Now suppose that X and Y are subshifts, m and a are nonnegative integers (standing for memory and anticipation), Φ is a function from the set of X-words of length $m+a+1$ into the symbol set for Y, and φ is a homomorphism from X to Y defined by the rule $\varphi(x)_i = \Phi(x_{i-m} \dots x_{i+a})$. The homomorphism φ is called a block code (a k-block code if $k = m+a+1$). The Curtis-Hedlund-Lyndon Theorem (trivial proof, fundamental observation) is that every homomorphism of subshifts is a block code.

If φ is a homomorphism of subshifts, and the domain and range of φ are the same subshift (X, σ), then φ is an *endomorphism* of (X, σ). Thus a one-dimensional cellular automaton map is an endomorphism of some full shift on n symbols.

A continuous map φ from a compact metric space X to itself is *positively expansive* if there exists $\epsilon > 0$ such that whenever x and x' are distinct points in X, there is a nonnegative integer k such that $\mathrm{dist}(\varphi^k(x), \varphi^k(x')) > \epsilon$. This property does not depend on the choice of metric compatible with the topology. Now if φ is an endomorphism of a one-sided subshift X and $k \in \mathbb{Z}_+$, then let $\hat{x}^{(k)}$ denote the sequence of words $[\varphi^i(x)_0 \dots \varphi^i(x)_k]$, $i = 0, 1, 2 \dots$. It is easy to check that φ is positively expansive if and only if there exists $k \in \mathbb{Z}_+$ such that the map $x \mapsto \hat{x}^{(k)}$ is injective.

A factor map φ between two-sided subshifts is *right-closing* if it never collapses distinct left-asymptotic points. This means that if $\varphi(x) = \varphi(x')$ and for some I it holds that $x_i = x'_i$ for $-\infty < i \leq I$, then $x = x'$. An easy compactness argument shows that φ being right-closing is equivalent to the following condition: there exists positive integers M, N such that for all x, x': if $x_i = x'_i$ for $-M < i \leq 0$, and $\varphi(x)_j = \varphi(x')_j$ for $0 \leq j \leq N$, then $x_1 = x'_1$. If $\varphi : \Sigma_A \to \Sigma_B$ and φ is a k-block code, then the condition can be stated with M fixed as k.

A factor map of one-sided subshifts, $X_A \to X_B$, is called *right-closing* if its defining block code defines a right-closing map of two-sided subshifts, $\Sigma_A \to \Sigma_B$. From the finite criterion of the previous paragraph we see that a factor map of one-sided subshifts is right-closing if and only if it is locally injective.

Left-closing factor maps are defined as above, with "right" replaced by "left". However, left closing does not mean locally injective on X_A (it would mean locally injective on sequences $\dots x_{-1}x_0$ with shift in the opposite direction). An important property of closing factor maps is that they are always finite-to-one. An endomorphism φ of an irreducible SFT is surjective if and only if it is finite-to-one and consequently every closing endomorphism is surjective.

For a thorough introduction to these topics, see [K2] or [LM].

3. Algebraic maps

In this section we consider factor maps which have an algebraic structure. This is the situation when the subshifts of finite type are also compact topological groups, the shift is a group automorphism and the factor map is a group homomorphism. An SFT which is also a topological group with the shift an automorphism is called a *Markov subgroup*. A result from [Ki1] shows that an irreducible Markov subgroup is topologically conjugate to a full shift, although the transition rules may be fairly complicated. We say a factor map between Markov subgroups which is also a group homomorphism is an *algebraic factor map*.

Example 3.1. Consider the full two-shift, $\{0,1\}^{\mathbf{Z}}$, as a group where the group operation is coordinate by coordinate addition, modulo two. The shift is clearly a group automorphism. Define φ by $\varphi(x)_i = x_i + x_{i+1}$ for all i. Then φ is an onto, two-to-one, group homomorphism.

Proposition 3.2. *Let $\varphi : \Sigma_A \to \Sigma_A$ be an algebraic factor map from an irreducible Markov subgroup to itself. Then there is a dense set of points in Σ_A which are periodic for both φ and the shift.*

Proof Let M be a positive integer such that no point of Σ_A has more than M preimages under φ. Fix any prime p with $p > M$. Then φ cannot map a point of least σ-period p to a point of lower period (for this would imply the entire σ-orbit of p points maps to a fixed point). It follows that for all $k > 0$, the kernel of φ^k contains no point with least σ-period equal to p.

We know that Σ_A is topologically conjugate to a full m-shift for some m, so $Fix_p(\Sigma_A)$ consists of $m^p - m$ points of least σ-period p and m σ-fixed points. Restricted to the subgroup $Fix_p(\Sigma_A)$, the homomorphism maps the fixed points to the fixed points and the points of period p to the points of period p. There is a power, k, of φ so that the image of $Fix_p(\Sigma_A)$ under φ^i is the same as the image under φ^k for all $i \geq k$. Therefore the points in the image of φ^k are φ-periodic. The cardinality of the kernel of φ^k on $Fix_p(\Sigma_A)$ is at most m, so at least $1/m$ of the points in $Fix_p(\Sigma_A)$ are φ-periodic.

Let $[i_1, \ldots, i_\ell]$ be any block which occurs in Σ_A. Since Σ_A is irreducible the block $[i_1, \ldots, i_\ell]$ will occur in more than $1/m$ of the σ-periodic points of all points of period p for any sufficiently large p. This means there is a jointly periodic point in the time zero cylinder set defined by $[i_1, \ldots, i_\ell]$ and so the jointly periodic points are dense in Σ_A. $\square$

Proposition 3.2 is a special case of a theorem in [KS] which states that the periodic points are dense in all transitive, d-dimensional Markov subgroups.

For certain algebraic maps φ, the φ-periods of points of a given σ-period are analyzed in [MOW]. These periods can be very different.

4. Closing maps

The following result is a pillar of our proof. (The essence of this result is due independently to Kurka [Ku] and Nasu [Na2]. We include an exposition in the last section of the paper.)

Lemma 4.1. [BFF] *Suppose ψ is a positively expansive map ψ which commutes with a mixing one-sided subshift of finite type. Then ψ is topologically conjugate to a mixing subshift of finite type.*

The closing property will let us exploit this characterization.

Lemma 4.2. *Suppose $\varphi : X_A \to X_A$ is a right-closing factor map from an irreducible, one-sided subshift of finite type to itself. Then for all sufficiently large N, the map $\sigma^N \varphi$ is positively expansive.*

Proof Suppose φ is a k-block map. Since φ is right-closing, if N is sufficiently large then for all x and for all $n \geq k-1$ the cylinder sets $[x_0, \ldots, x_n]$ and $[\varphi(x)_0, \ldots, \varphi(x)_{n+N}]$ determine x_{n+1}. To a point $x \in X_A$ assign the sequence of $k+N-1$ blocks $[(\sigma^N\varphi)^i(x)_0, \ldots, (\sigma^N\varphi)^i(x)_{N+k-2}]$, $i \geq 0$. To show $\sigma^n\varphi$ is positively expansive, it suffices to show this sequence of blocks determines x.

To see this observe that the block $[x_0, \dots, x_{N+k-2}]$ determines the block $[\varphi(x)_0, \dots, \varphi(x)_{N-1}]$ and the block $[\sigma^N\varphi(x)_0, \dots, \sigma^N\varphi(x)_{N+k-2}]$ is the same as $[\varphi(x)_N, \dots, \varphi(x)_{2N+k-2}]$. This means we have the blocks $[x_0, \dots, x_{N+k-2}]$ and $[\varphi(x)_0, \dots, \varphi(x)_{2N+k-2}]$ which together determine x_{N+k-1}. Likewise, the blocks for $i = 1$ and 2 determine $\varphi(x)_{2N+k-1}$ which together with what we already have determines x_{N+k}. Continuing in this manner we see that x is completely determined. □

Here is the one-sided version of our main result.

Theorem 4.3. *Suppose $\varphi : X_A \to X_A$ is a right-closing factor map from a mixing one-sided subshift of finite type to itself. Then the points which are jointly periodic for σ and φ are dense in X_A.*

Proof Appealing to Lemmas 4.1 and 4.2, we choose a positive integer N such that $\sigma^N\varphi$ is topologically conjugate to a mixing subshift of finite type. The $\sigma^N\varphi$-periodic points are dense in X_A. We will show these points are jointly periodic for σ and φ.

First we claim the two maps $\sigma^N\varphi$ and σ have the same preperiodic points. Every σ-preperiodic point is a $\sigma^N\varphi$-preperiodic point because for each ℓ and p the points $x \in X_A$ with $\sigma^{\ell+p}(x) = \sigma^p(x)$ form a finite, $\sigma^N\varphi$-invariant set. Similarly, every $\sigma^N\varphi$-preperiodic point is a σ-preperiodic point.

Next we show the $\sigma^N\varphi$-periodic points are σ-periodic. Suppose $x \in X_A$ is such that $(\sigma^N\varphi)^p(x) = x$. Because x must be σ-preperiodic, there are ℓ and q such that $\sigma^{\ell p}(x)$ has σ-period q. Therefore $\sigma^{(N-1)\ell p}\varphi^{\ell p}(\sigma^{\ell p}(x))$ is also a fixed point of σ^q. But $\sigma^{(N-1)\ell p}\varphi^{\ell p}(\sigma^{\ell p}(x)) = (\sigma^N\varphi)^{\ell p}(x) = x$.

Finally we show the $\sigma^N\varphi$-periodic points are φ-periodic. If $x \in X_A$ has $\sigma^N\varphi$-period p, then it has σ-period q for some q, and therefore $\varphi^{pq}(x) = \varphi^{pq}\sigma^{Npq}(x) = (\varphi\sigma^N)^{pq}(x) = x$. □

It is now an easy reduction to obtain our main result, the two-sided version of Theorem 4.3.

Theorem 4.4. *Suppose $\varphi : \Sigma_A \to \Sigma_A$ is a right or left-closing factor map from a mixing subshift of finite type to itself. Then the points which are jointly periodic for σ and φ are dense in Σ_A.*

Proof Suppose $\varphi : \Sigma_A \to \Sigma_A$ is a right-closing factor map with anticipation a and memory m. Then $\sigma^m\varphi$ is a right-closing factor map with no memory and with anticipation $a + m$. We can use $\sigma^m\varphi$ to define a right-closing factor map from the one-sided subshift of finite type X_A to itself. By Theorem 4.3, the points which are jointly periodic for σ and $\sigma^m\varphi$ are dense in X_A.

Since (Σ_A, σ) is the natural extension or inverse limit of σ acting on X_A and the jointly periodic points for σ and $\sigma^m\varphi$ are dense in X_A the resulting points which are jointly periodic for σ and $\sigma^m\varphi$ in Σ_A are dense. Applying the reasoning used in the proof of Theorem 4.3 we conclude that the points which are jointly periodic for σ and φ are dense in Σ_A.

If φ is left-closing with respect to σ_A, then φ is right-closing with respect to $(\sigma_A)^{-1}$ and we may apply the right-closing result. □

5. Examples of closing maps

Most cellular automata are not closing maps, but many are. For example, all automorphisms are closing maps. Constructions of Ashley [A] yield noninjective closing endomorphisms of mixing shifts of finite type (and in particular closing cellular automata) with a rich range of behavior on subsystems.

The permutive maps of Hedlund [H] are a large and accessible class of closing maps. Because they can be analyzed very easily, we include a brief discussion. Let φ be a one-sided k-block cellular automaton map (that is, an endomorphism of a one-sided full shift) with $k > 1$. Suppose φ is *right permutive*: if $x_1 \dots x_{k-1} = x'_1 \dots x'_{k-1}$ and $x_k \neq x'_k$, then $\varphi(x)_1 \neq \varphi(x')_1$. It is clear that φ is positively expansive and so by lemma 4.1 φ is topologically conjugate to an SFT. A conjugacy can also be displayed directly. Define a one-sided SFT (X_φ, σ) as follows. The symbols of X_φ are the $(k-1)$-blocks of X_A. Define transitions by saying $[i_1, \dots, i_{k-1}]$ can be followed by $[j_1, \dots, j_{k-1}]$ when there is a block $[i'_1, \dots, i'_{k-1}]$ so that $\varphi([i_1, \dots, i_{k-1}, i'_1, \dots, i'_{k-1}]) = [j_1, \dots, j_{k-1}]$. To a point x in X_A associate the sequence $\bar{x} = \bar{x}_0, \bar{x}_1, \dots$ where $\bar{x}_i$ is the word $\varphi^i(x)_0 \dots \varphi^i(x)_{k-2}$. Then it is not difficult to check that the rule $x \mapsto \bar{x}$ defines a topological conjugacy from (X_A, φ) to (X_φ, σ).

Lemma 4.2 shows that a right-closing map composed with a high enough power of the shift is positively expansive and we just saw that a k-block, right-permutive map is itself positively expansive when $k > 1$. The *multiplication cellular automata* studied by F. Blanchard and A. Maass [BM] are nontrivial natural examples of right-closing maps and many of them are not positively expansive. Given positive integers k and n greater than 1, with k dividing n, the multiplication c.a. φ is the endomorphism of the one-sided n-shift which expresses multiplication by k (modulo 1) in base n. Blanchard and Maass showed this map is right-closing and will be positively expansive if and only if every prime dividing n also divides k. We give an example (with an explanation pointed out to us independently by F. Blanchard and U. Fiebig).

Example 5.1. View X_{10} as the set of one-sided infinite sequences obtained by expressing the real numbers in the unit interval as decimals in base ten. Then define the right-closing factor map from X_{10} to itself using multiplication by two, as real numbers, on these sequences. Consider a rational number with a power of ten as the denominator. It has two expansions. For example, $000100\dots$ and $0000999\dots$. Multiplying by two gives $000200\dots$ and $000199\dots$. Multiplying again gives $000400\dots$ and $000399\dots$. Continuing, we see that the two sequences always agree in the first three coordinates. All rational numbers with a power of ten as the denominator has two such representations and so this map on X_{10} is not positively expansive.

6. Closing arguments ...

The purpose of this section is to provide some background proofs and facts involving closing maps, and to explain how some of these facts become more transparent (for us, at least) if viewed in terms of resolving maps.

Let $\varphi : X_A \to X_B$ be a one-block factor map between two irreducible one-sided subshifts of finite type. Consider the following conditions on a map on symbols (also called φ):

(1) (Existence) If $\varphi(a) = b$ and b' follows b in X_B, then there exists a symbol a' such that a' follows a and $\varphi(a') = b'$.
(2) (Uniqueness) If $\varphi(a) = b$ and b' follows b in X_B, then there is at most one symbol a' such that a' follows a and $\varphi(a') = b'$.

A one-block factor map between irreducible SFT's satisfying conditions (1) and (2) is called *right-resolving*. A right-resolving factor map is clearly finite-to-one. It follows from condition (1) that a right-resolving map is locally injective and from condition (2) that it is an open map. So, a right-resolving factor map between two one-sided SFT's is a local homeomorphism. On the other hand, when φ is a finite-to-one factor map between irreducible SFT's, it is a consequence of the Perron-Frobenius theorem that conditions (1) and (2) are equivalent. (See [LM] Prop. 8.2.2 for (2) $\Rightarrow$ (1); the converse is similar.) There is of course a similar definition of left-resolving. The resolving maps have played a central role in the coding theory of symbolic dynamics ([K2],[LM]).

A right-resolving map is clearly right-closing and modulo a recoding the converse is true. It is a standard result in symbolic dynamics ([K2] Prop. 4.3.3) which we formulate in the following lemma.

Lemma 6.1. *Suppose $\varphi : X_A \to X_B$ is a right-closing factor map between one-sided irreducible subshifts of finite type, then there is an irreducible subshift of finite type X_C, a right-resolving factor map $\psi : X_C \to X_B$ and a topological conjugacy $\alpha : X_A \to X_C$ such that $\varphi = \alpha \circ \psi$.*

Lemma 6.2. *Suppose $\varphi : X_A \to X_B$ is a factor map between one-sided irreducible subshifts of finite type then φ is right-closing if and only if it finite-to-one and open.*

Proof A right-closing map is finite-to-one and by lemma 6.1 is open.

Suppose φ is a finite-to-one and open. An easy compactness argument shows this is equivalent to the following uniform existence condition.

- There exists $N > 0$ such that for all x, y: if $\varphi(x)_i = y_i$ for $0 \le i \le N$ then there exists x' such that $x'_0 = x_0$ and $\varphi(x) = y$.

A recoding argument similar to the one used to prove lemma 6.1 can be used to show that any map satisfying the uniform existence condition can be recoded to satisfy condition 1 (Existence) in the definition of right-resolving. Since we have also assumed the map is finite-to-one the recoded map will be right-resolving and so the original map was right-closing. (This was done explicitly in [BT], Prop. 5.1.) $\square$

The characterization above, well known in symbolic dynamics, will make some topological properties of closing maps obvious. (Note, though, in the case $X_A = X_B$ we have not produced a topological conjugacy of endomorphisms: as iterated maps, the maps φ and ψ above may be quite different.)

Lemma 6.3. *Let $\varphi : X_A \to X_B$ be a finite-to-one factor map between two irreducible one-sided subshifts of finite type. The following are equivalent.*

(1) *φ is right-closing (i.e. locally injective on X_A)*
(2) *φ is an open mapping*
(3) *φ is a local homeomorphism*

Proof Clearly a local homeomorphism is locally injective and open. Conversely, if φ is right-closing or open, then by lemma 6.1 and lemma 6.2 it is a homeomorphism followed by a local homeomorphism, so it is a local homeomorphism. $\square$

Lemma 6.4. [Pa] *Suppose X is a one-sided subshift. Then it is a subshift of finite type if and only if σ is an open mapping.*

Proof Suppose (X, σ) is a k-step SFT. If $[i_0, \dots, i_\ell]$ is a time zero cylinder set with $\ell \geq k$, then σ maps it onto the time zero cylinder set $[i_1, \dots, i_\ell]$ and so σ is an open mapping.

Suppose σ is an open mapping. Then σ of any cylinder set is open and compact and so is a finite union of cylinder sets. There is a k so that σ of every time zero, length one cylinder set is a union of time zero, length k cylinder sets. This means σ of a time t, length ℓ cylinder set is a union of time t, length $\ell + k$ cylinder sets and (X, σ) is a k-step SFT. □

Lemma 6.5. [Ku] *Suppose φ is a positively expansive endomorphism of a one-sided subshift. Then φ is right-closing.*

Proof Let $N > 0$ be such that $x = x'$ whenever $\varphi^i(x)_k = \varphi^i(x')_k$ for $0 \leq k \leq N$ for all $i \geq 0$. Then the restriction of φ to any cylinder of the form $\{x : x_0 \dots x_N = w_0 \dots w_n\}$ must be injective. □

Lemma 6.6. *Suppose $\varphi : X_A \to X_A$ is a factor map from an irreducible one-sided subshift of finite type to itself. Then (X_A, φ) is topologically conjugate to a one-sided subshift of finite type if and only if φ is positively expansive.*

Proof Suppose φ is a positively expansive. By lemma 6.5 φ is right-closing, and then by lemma 6.3 φ is open. Thus φ is conjugate to a subshift which is an open map, and by lemma 6.4 this subshift must be of finite type.

The other direction is trivial, because conjugacy respects positive expansiveness and every subshift is positively expansive. □

Lemma 6.6 is due independently to Nasu (who proved it [Na2]) and Kurka (who in a special case gave an argument which works in general [Ku]). Lemma 6.6 is false if the hypothesis of irreducibility is dropped [BFF]. The analogous question for two-sided subshifts is an important open question of Nasu [Na1]: must an expansive automorphism of an irreducible SFT be topologically conjugate to an SFT?

Notice, the property that a factor map is right-closing does not change under composition with powers of the shift. So, if $\sigma^n \varphi$ is topologically conjugate to a one-sided irreducible SFT, then φ must be right-closing. Thus our proof of Theorem 4.4 can only work for right-closing maps φ.

Proof of Lemma 4.1 We are given a positively expansive map ψ which commutes with some one-sided mixing SFT (X_A, σ_A). By Lemma 6.6, there is a conjugacy of ψ to some one-sided SFT (X_B, σ_B). This conjugacy conjugates σ_A to some mixing endomorphism φ of (X_B, σ_B). Following [BFF], we will show σ_B is mixing. Suppose it is not, then (perhaps after passing to a power of ψ) (X_B, σ_B) has more than one irreducible component. Since φ permutes the irreducible components of σ_B, we may choose N such that φ^N maps each irreducible component of X_B to itself. Let x be a point in some terminal component C and let x' be a point in some other component C'. Because φ is mixing, for some $k > 0$ there is a point z such that $z_0 = x_0$ and $(\phi^{kN} z)_0 = x'_0$. The point z can only be contained in C, and the σ_C-periodic points are dense in C, so we can take z to be periodic. But then ϕ^{kN} sends a periodic point of C to a periodic point which must lie in C', and this is contradiction. □

If in Lemma 4.1 it is only assumed that σ_A is irreducible, rather than mixing, then one can only conclude that ψ is conjugate to a disjoint union of irreducible subshifts of finite type (that is, an SFT with dense periodic points).

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